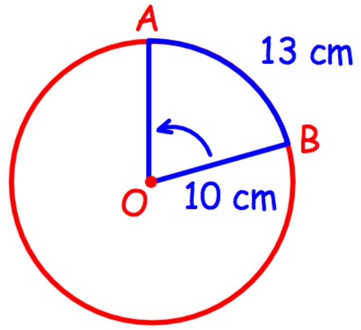
TRIGONOMETRIC FUNCTIONS

RADIAN MEASURE: given by the ratio $\frac{\text{arc } AB}{r}$

$$\frac{\text{arc } AB}{r} = \frac{13}{10} = 1.3$$

$$\angle AOB = 1.3 \text{ rads}$$



The ratio for a complete turn is 2π, so

$$180° = \pi \text{ radians}$$

RADS → DEGREES: $\div \pi \times 180$

DEGREES → RADS: $\div 180 \times \pi$

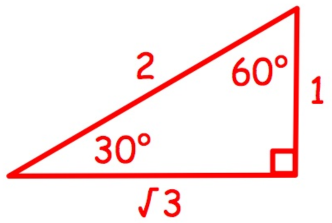
(1) $1.3 \text{ rads} = \frac{1.3}{\pi} \times 180° = 74.484\ldots \approx 74.5°$

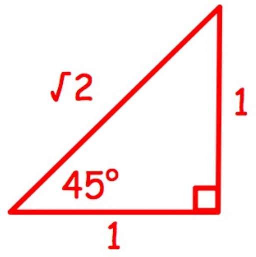
(2) $140° = \frac{140}{180} \times \pi = 2.4434\ldots \approx 2.44 \text{ rads}$

as multiples of π

(3) $260°$ $\quad \frac{260}{180}\pi = \frac{13}{9}\pi = \frac{13\pi}{9}$

EXACT VALUES:





30°	45°	60°	90°
$^{\pi}/_{6}$	$^{\pi}/_{4}$	$^{\pi}/_{3}$	$^{\pi}/_{2}$

multiples:

(1) $^{5\pi}/_{3}$ $\quad 5 \times {}^{\pi}/_{3} = 5 \times 60° = 300°$

(2) 210° $\quad 7 \times 30° = 7 \times {}^{\pi}/_{6} = {}^{7\pi}/_{6}$

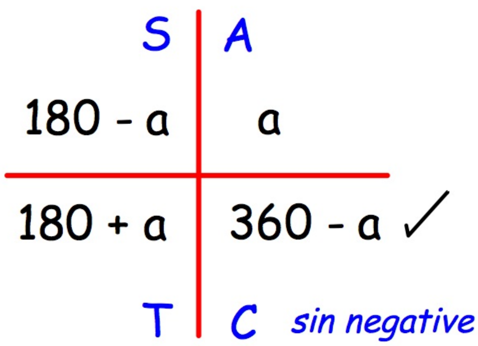
beyond acute angles:

(1) $\sin 300°$

$= \sin (360 - 60)°$

$= -\sin 60°$

$= -\,{}^{\sqrt{3}}/_{2}$

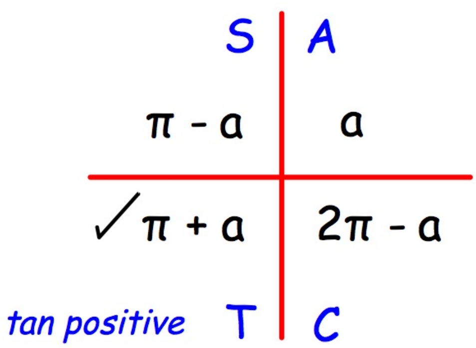


(2) $\tan {}^{7\pi}/_{6}$

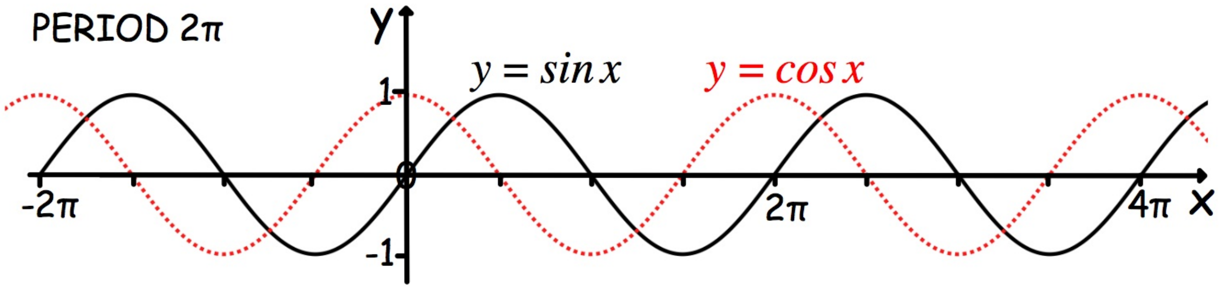
$= \tan (\pi + {}^{\pi}/_{6})$

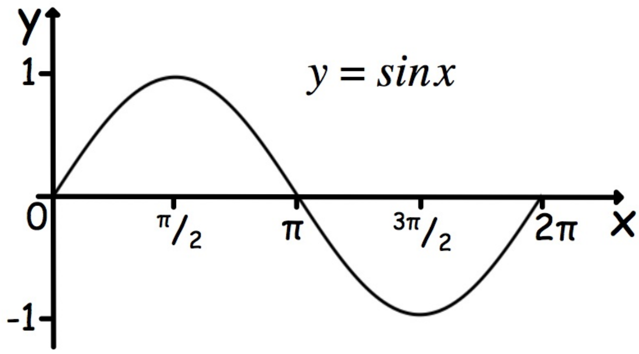
$= +\tan {}^{\pi}/_{6}$

$= {}^{1}/_{\sqrt{3}}$

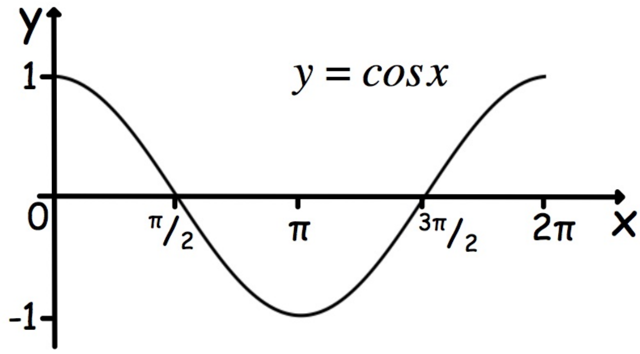


TRIGONOMETRIC GRAPHS





Max TP $(\pi/2, 1)$
Min TP $(3\pi/2, -1)$

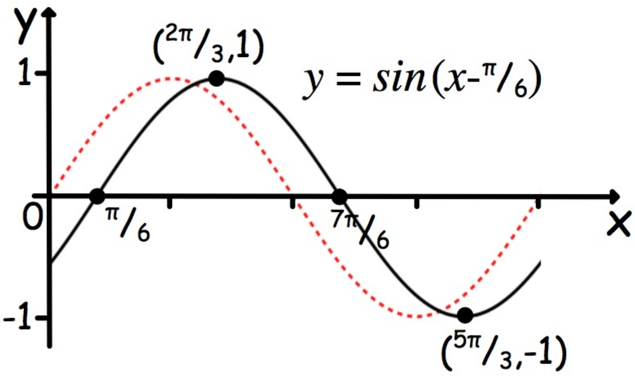


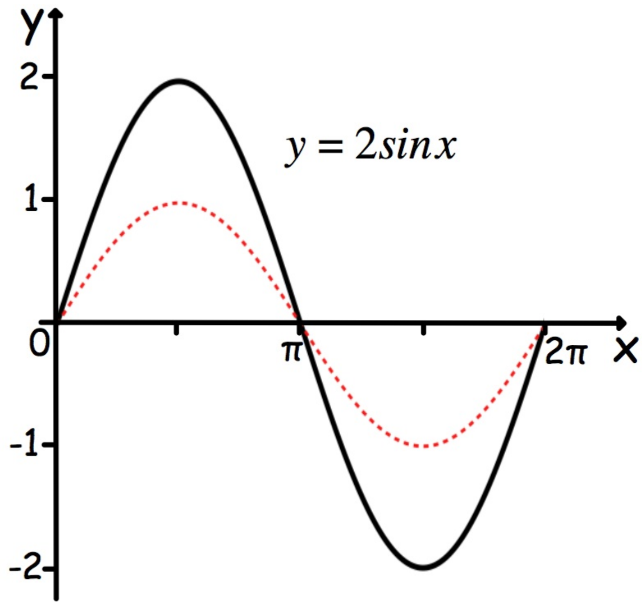
Max TP $(0, 1)$
Min TP $(\pi, -1)$

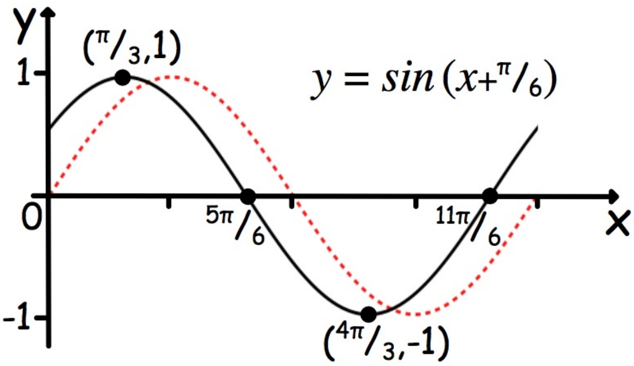
TRANSFORMATIONS

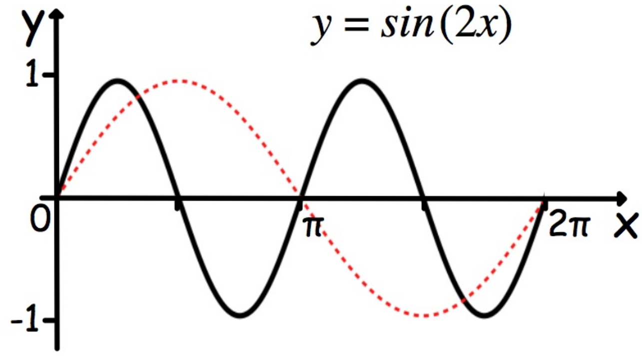
$y = a\sin x$	stretch a units vertically
$y = \sin(bx)$	period $2\pi \div b$
$y = \sin(x + c)$	shift $-c$ rads horizontally
$y = \sin x + d$	shift d units vertically

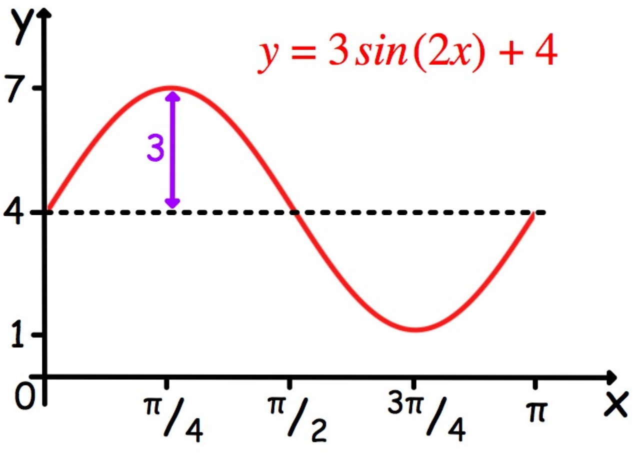
TRANSFORMATIONS:

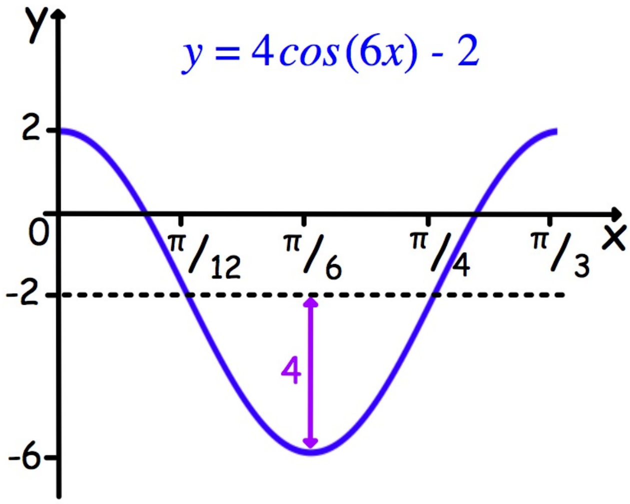












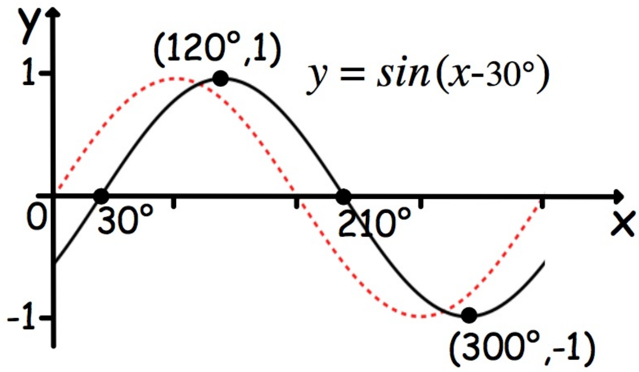
$(\pi/2\ ,\ 1)$ → ÷2, ×3 + 4 → $(\pi/4\ ,\ 7)$ MAX. TP

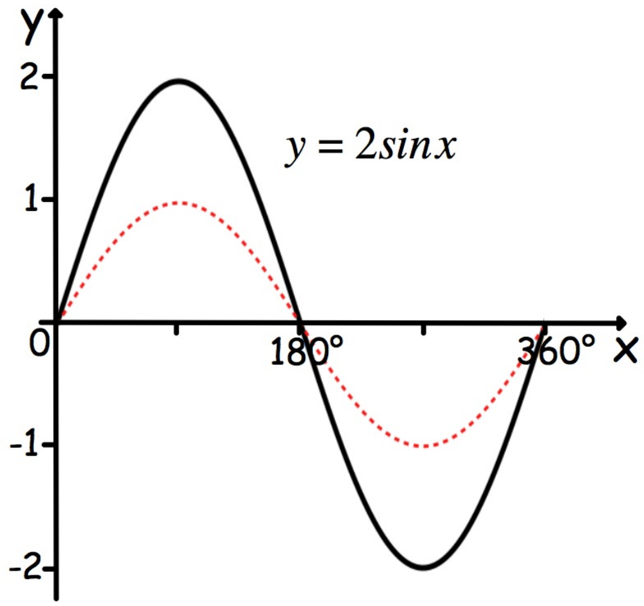
$(3\pi/2\ ,\ -1)$ → ÷2, ×3 + 4 → $(3\pi/4\ ,\ 1)$ MIN. TP

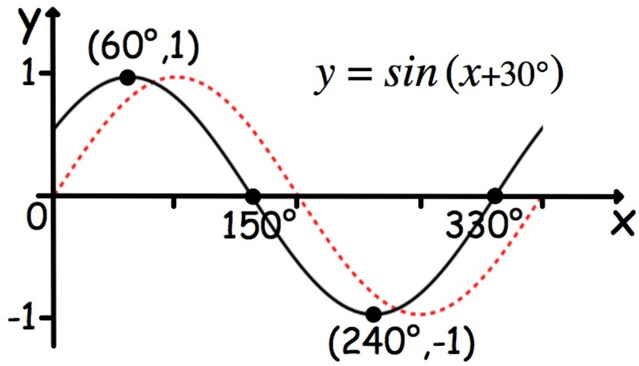
$(0\ ,\ 1)$ → ÷6, ×4 - 2 → $(0\ ,\ 2)$ MAX. TP

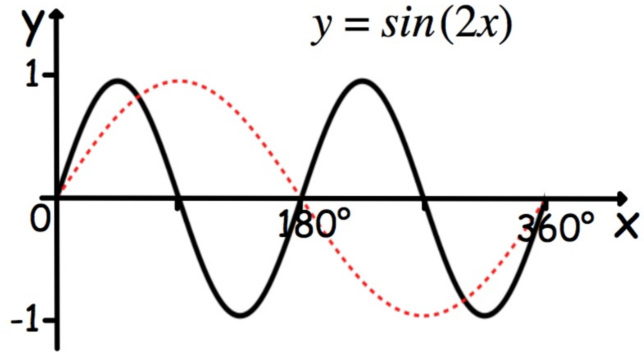
$(\pi\ ,\ -1)$ → ÷6, ×4 - 2 → $(\pi/6\ ,\ -6)$ MIN. TP

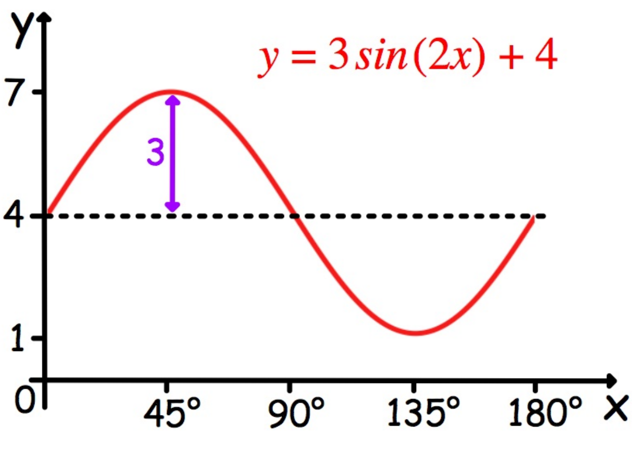
TRANSFORMATIONS:

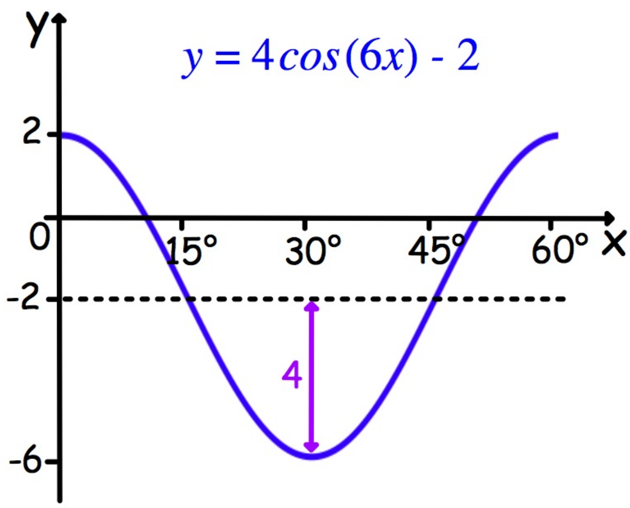












(90°, 1)	(270°, -1)	(0°, 1)	(180°, -1)
÷2 ×3 + 4	÷2 ×3 + 4	÷6 ×4 - 2	÷6 ×4 - 2
(45°, 7)	(135°, 1)	(0°, 2)	(30°, -6)
MAX. TP	MIN. TP	MAX. TP	MIN. TP

TRIGONOMETRIC EQUATIONS

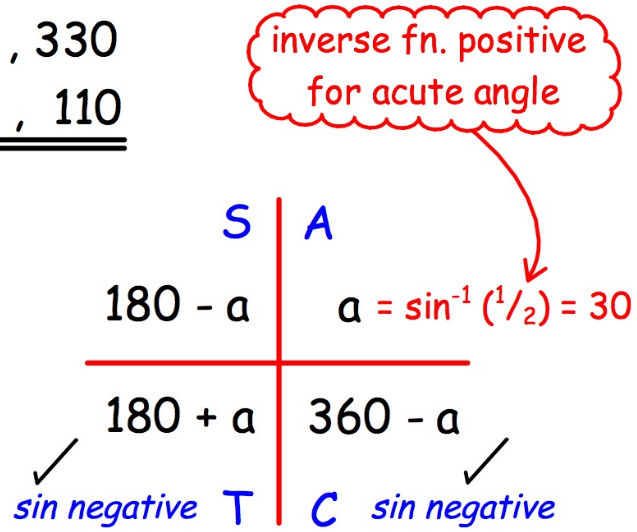
(1) $2\sin 3x^\circ + 6 = 5$, $0 \le x \le 120$

$2\sin 3x^\circ = -1$

$\sin 3x^\circ = -\,^1/_2$

$3x = 210 , 330$

$x = 70 , 110$



(2) $\sin 3x = -\,^1/_2$, $0 \le x \le \,^{2\pi}/_3$

$3x = \,^{7\pi}/_6 , \,^{11\pi}/_6$

$x = \,^{7\pi}/_{18} , \,^{11\pi}/_{18}$

MULTIPLE ANGLES: solving over more than one period

(3) $\sin 3x^\circ = -\,^1/_2$, $0 \le x \le 360$

$3x = 210 , 330$

$x = 70 , 110$

add multiples of period 120°

$x = 70 , 110 , 190 , 230 , 310 , 350$

USING GRAPHS: for 0°, 90°, 180°, 270°, 360°

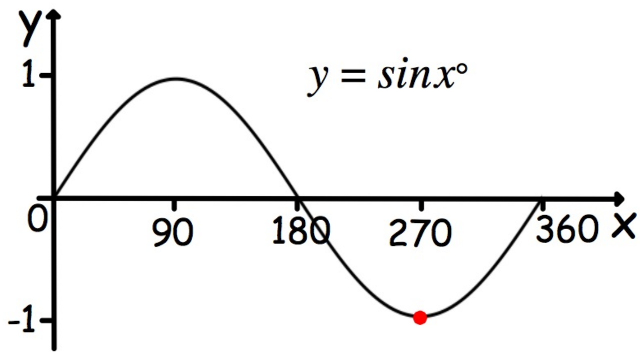
(4) $\sin 2x° = -1$, $0 \le x \le 360$

$2x = 270$

$x = 135$

add multiples of period 180°

$x = 135 , 315$



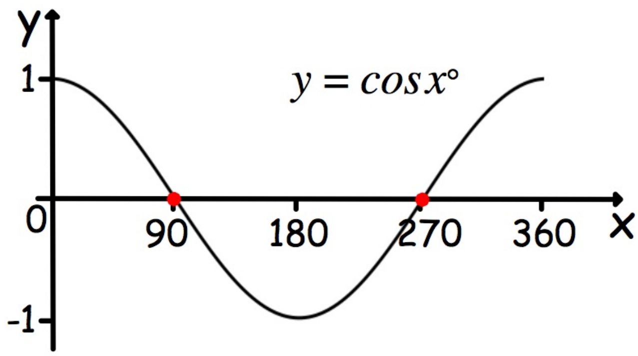
(5) $\cos 2x° = 0$, $0 \le x \le 360$

$2x = 90 , 270$

$x = 45 , 135$

add multiples of period 180°

$x = 45 , 135 , 225 , 315$



SEQUENCES

Sequences can be generated by rules:

(i) n^{th} term formula

relates terms to the natural numbers
the formula can generate any term required

(ii) recurrence relation

relates one term to the next
requires an initial term
work through sequence to generate term required

1 , 2 , 4 , 8 , 16 ...

n^{th} term formula $\quad U_n = 2^{n-1}$

5th term $\quad U_5 = 2^{5-1} = 2^4 = 16$

recurrence relation $\quad U_{n+1} = 2U_n \quad , \quad U_1 = 1$

$U_2 = 2U_1 = 2 \times 1 = 2$

$U_3 = 2U_2 = 2 \times 2 = 4$

$U_4 = 2U_3 = 2 \times 4 = 8$

$U_5 = 2U_4 = 2 \times 8 = 16$

LINEAR RECURRENCE RELATIONS

$$U_{n+1} = aU_n + b$$

initial term often U_0 , "U nought"

NOTE: $U_n = aU_{n-1} + b$ is the same rule

A park has 500 of a pest species which increases by 20% each year, so 200 are removed each year.

How many years to eliminate the pest?

$U_{n+1} = 1.2\,U_n - 200$, $U_0 = 500$

Growth Factor
100% + 20% = 120%

$U_0 = 500$

$U_1 = 1.2 \times 500 - 200 = 400$

$U_2 = 1.2 \times 400 - 200 = 280$

$U_3 = 1.2 \times 280 - 200 = 136$

$U_4 = 1.2 \times 136 - 200 = -36.8 < 0$

4 years required

A recurrence relation is of the form $U_{n+1} = aU_n + b$ where $U_{10} = 2$, $U_{11} = 5$ and $U_{12} = 17$.

Find (i) a and b (ii) U_{13} (iii) U_9

(i) $U_{n+1} = aU_n + b$

$U_{12} = aU_{11} + b$

$17 = a \times 5 + b$

$U_{11} = aU_{10} + b$

$5 = a \times 2 + b$

solve simultaneous equations

$5a + b = 17$

$2a + b = 5$

subtract

$3a = 12$

$a = 4$

$2a + b = 5$

$8 + b = 5$

$b = -3$

$U_{n+1} = aU_n + b$

$U_{n+1} = 4U_n - 3$

(ii) $U_{n+1} = 4U_n - 3$

$U_{13} = 4U_{12} - 3$

$U_{13} = 4 \times 17 - 3$

$U_{13} = 65$

(iii) $U_{n+1} = 4U_n - 3$

$U_{10} = 4U_9 - 3$

$2 = 4U_9 - 3$

$5 = 4U_9$

$U_9 = 5/4$

LIMITS

Some sequences converge to a particular value.

n^{th} term formula

1 , 2 , 4 , 8 , 16 .... for $U_n = 2^{n-1}$

heading to infinity (∞) as $n \to \infty$, $U_n \to \infty$

1 , $^1/_2$, $^1/_4$, $^1/_8$, $^1/_{16}$ .... for $U_n = (^1/_2)^{n-1}$

heading to the limit zero as $n \to \infty$, $U_n \to 0$

A sequence is given by the formula $U_n = \dfrac{3n + 1}{n}$

(a) Find the first 3 terms of the sequence.

(b) Find the limit of the sequence.

(a) $U_1 = \dfrac{3 \times 1 + 1}{1} = 4$

$U_2 = \dfrac{3 \times 2 + 1}{2} = 3^1/_2$

$U_3 = \dfrac{3 \times 3 + 1}{3} = 3^1/_3$

(b) choose very large values:

$U_{1000} = \dfrac{3 \times 1000 + 1}{1000} = 3.001$

$U_{10000} = \dfrac{3 \times 10000 + 1}{10000} = 3.0001$

as $n \to \infty$, $U_n \to 3$

LIMIT is 3

recurrence relation $U_{n+1} = aU_n + b$

has a limit if $-1 < a < 1$ ie. a fraction

for limit L , $L = aL + b$

The sequence converges on the limit regardless of U_o

$U_{n+1} = 0.2U_n + 4$

$U_o = 4$; 4 , 4.8 , 4.96 , 4.992 , 4.994 , 4.99968 ...

$U_o = 6$; 6 , 5.2 , 5.04 , 5.008 , 5.0016 , 5.00032 ...

$U_o = 5$; 5 , 5 , 5 , 5 , 5 , 5 ...

start with the limit 5 and next term is the limit.

for $U_{n+1} = 0.2U_n + 4$

as $n \longrightarrow \infty$, $U_n \longrightarrow 5$

LIMIT is 5

can generalise the result:

$$L = aL + b$$

$$L - aL = b$$

$$L(1 - a) = b$$

$$L = \frac{b}{1 - a}$$

(1) In a park, each day 80% of litter is cleared but 4 kg of new litter is dropped. Find the amount of litter in the long-term.

80% removed, so 20% remains

$U_{n+1} = 0.2U_n + 4$

has a limit since $-1 < 0.2 < 1$

$$L = 0.2L + 4$$
$$0.8L = 4$$
$$L = 4 \div 0.8$$
$$L = 5$$

check:

$0.2 \times 5 = 1$ Min. level

$1 + 4 = 5$ Max. level

In the long-term the litter settles at 5 kg.

(2) A car leaks 25% of its oil each day so 1 litre of oil is added daily. The oil level must never fall below 4 litres. Is it safe in the long-term?

25% removed, so 75% remains

$U_{n+1} = 0.75U_n + 1$

has a limit since $-1 < 0.75 < 1$

$$L = 0.75L + 1$$
$$0.25L = 1$$
$$L = 1 \div 0.25$$
$$L = 4$$

check:

$0.75 \times 4 = 3$ Min. level

$3 + 1 = 4$ Max. level

NOT safe in the long-term as the level falls to 3 litres before topping-up.