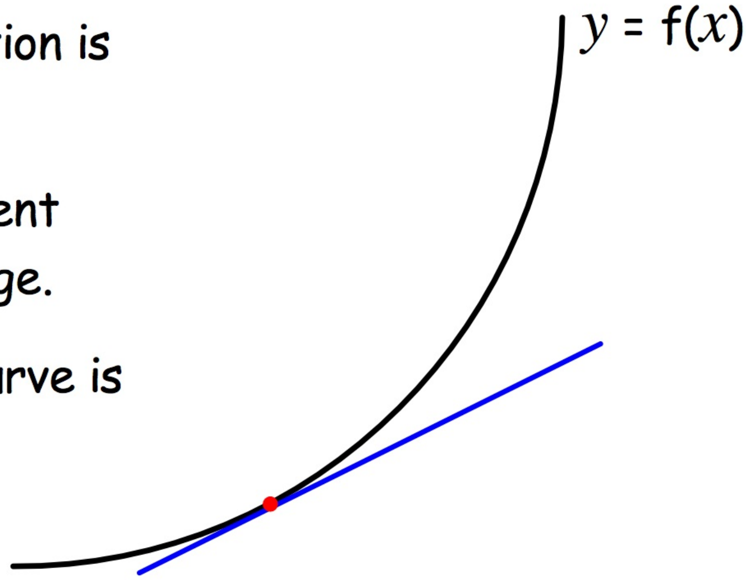
DIFFERENTIAL CALCULUS

Examines the effect of change on a function.
A change in the domain produces a change in the range.

The value of the function is continuously changing.

At any point the gradient gives the rate of change.

The gradient of the curve is given by the tangent.



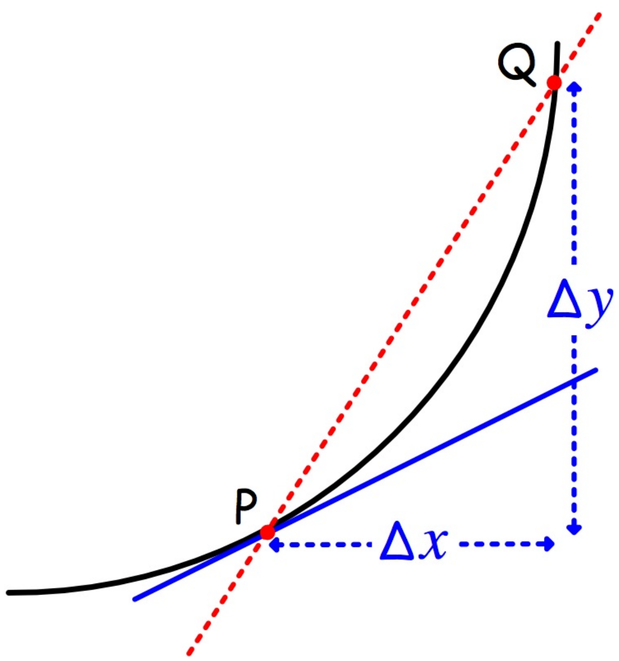
The gradient of line PQ is $\Delta y / \Delta x$

and as Q moves closer to P , $\Delta x \to 0$

The gradient at P is ,

$$\frac{\Delta y}{\Delta x} \quad \text{as} \quad \Delta x \to 0$$

written $\dfrac{dy}{dx}$



RULE: variety of notations

$y = x^n$ $\qquad$ $f(x) = x^n$

$\frac{dy}{dx} = nx^{n-1}$ $\qquad$ $f'(x) = nx^{n-1}$ $\qquad$ $\frac{d}{dx}(x^n) = nx^{n-1}$

TERMINOLOGY:

gradient of curve/tangent

rate of change of a function
(speed , acceleration)

derived function
derivative

differential
differentiation

OTHER RULES:

$F(x) = k\,f(x)$ $\qquad$ $F(x) = f(x) + g(x)$

$F'(x) = k\,f'(x)$ $\qquad$ $F'(x) = f'(x) + g'(x)$

(1) $f(x) = x^3 - x^2 + x + 5$

$f'(x) = 3x^2 - 2x + 1$

differentiates to 0

(2) $g(u) = 2u^5 - 6u^3 + 7u - 3$

$g'(u) = 10u^4 - 18u^2 + 7$

(3) Find the gradient of the curve $y = x^2 - 3x + 2$ at the point where $x = 4$.

$f(x) = x^2 - 3x + 2$

$f'(x) = 2x - 3$

$f'(4) = 2 \times 4 - 3 = 5$

gradient 5

INDICES

Use indices rules to express terms in the form x^n.

$a^m \times a^n = a^{m+n}$ $\qquad \frac{1}{a^p} = a^{-p}$ $\qquad \sqrt[n]{a^m} = a^{\frac{m}{n}}$

$a^m \div a^n = a^{m-n}$

(1) $f(x) = \frac{3}{x^2}$

$= 3x^{-2}$

$f'(x) = -6x^{-3}$

$= -\frac{6}{x^3}$

(2) $g(x) = \frac{1}{3x}$

$= \frac{1}{3}x^{-1}$

$g'(x) = -\frac{1}{3}x^{-2}$

$= -\frac{1}{3x^2}$

(3) $h(x) = \sqrt{x^3}$, find $h'(9)$

$h(x) = x^{\frac{3}{2}}$

$h'(x) = \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{3}{2}\sqrt{x}$

$h'(9) = \frac{3}{2}\sqrt{9}$

$= \frac{9}{2}$

(4) $k(x) = \frac{1}{\sqrt{x}}$, find $k'(4)$

$k(x) = x^{-\frac{1}{2}}$

$k'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$

$= -\frac{1}{2\sqrt{x^3}}$

$k'(4) = -\frac{1}{2\sqrt{4^3}}$

($\sqrt{4}$ and cube)

$= -\frac{1}{16}$

BRACKETS AND QUOTIENTS

Differentiate sums and differences of terms x^n, so 'break' brackets and 'split' quotients.

(1) $f(x) = \dfrac{(x+1)^2}{x}$

$$\frac{x^2 + 2x + 1}{x}$$

$$= \frac{x^2}{x} + \frac{2x}{x} + \frac{1}{x}$$

$$= x + 2 + x^{-1}$$

$f(x) = x + 2 + x^{-1}$

$f'(x) = 1 + 0 - x^{-2}$

$= 1 - \dfrac{1}{x^2}$

(2) $f(x) = \dfrac{x^2 + 1}{\sqrt{x}}$

$$\frac{x^2}{x^{1/2}} + \frac{1}{x^{1/2}}$$

$$= x^{3/2} + x^{-1/2}$$

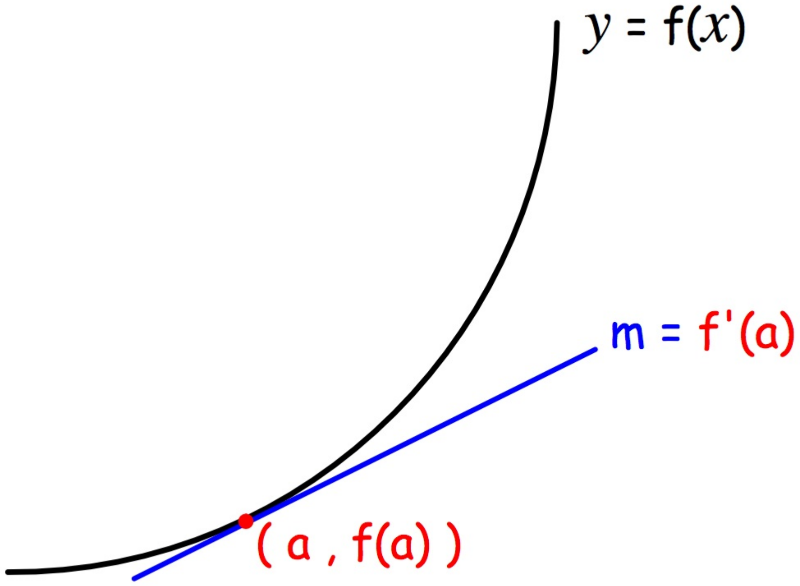
$f(x) = x^{3/2} + x^{-1/2}$

$f'(x) = \dfrac{3}{2}x^{1/2} - \dfrac{1}{2}x^{-3/2}$

$= \dfrac{3}{2}\sqrt{x} - \dfrac{1}{2\sqrt{x^3}}$

EQUATION OF A TANGENT

The gradient of the curve at some point is given by the tangent to the curve at that point.



Find the equation of the tangent to the curve $y = x^2 - x$ at the point where $x = 3$.

$f(x) = x^2 - x$

$f(3) = 3^2 - 3 = 6$

point $(3,6)$

$f(x) = x^2 - x$

$f'(x) = 2x - 1$

$f'(3) = 2 \times 3 - 1 = 5$

gradient 5

a b
(3 , 6)
m = 5

$$y - b = m(x - a)$$

$$y - 6 = 5(x - 3)$$

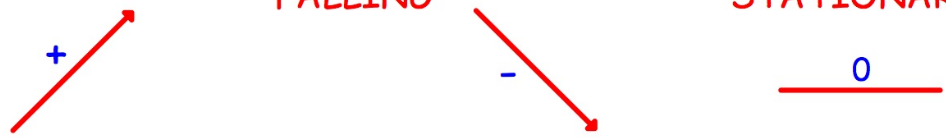
$$y - 6 = 5x - 15$$

$$\underline{\underline{y = 5x - 9}}$$

GRAPH OF THE DERIVED FUNCTION , $y = f'(x)$

The graph of the gradient.

Graph RISING — FALLING — STATIONARY



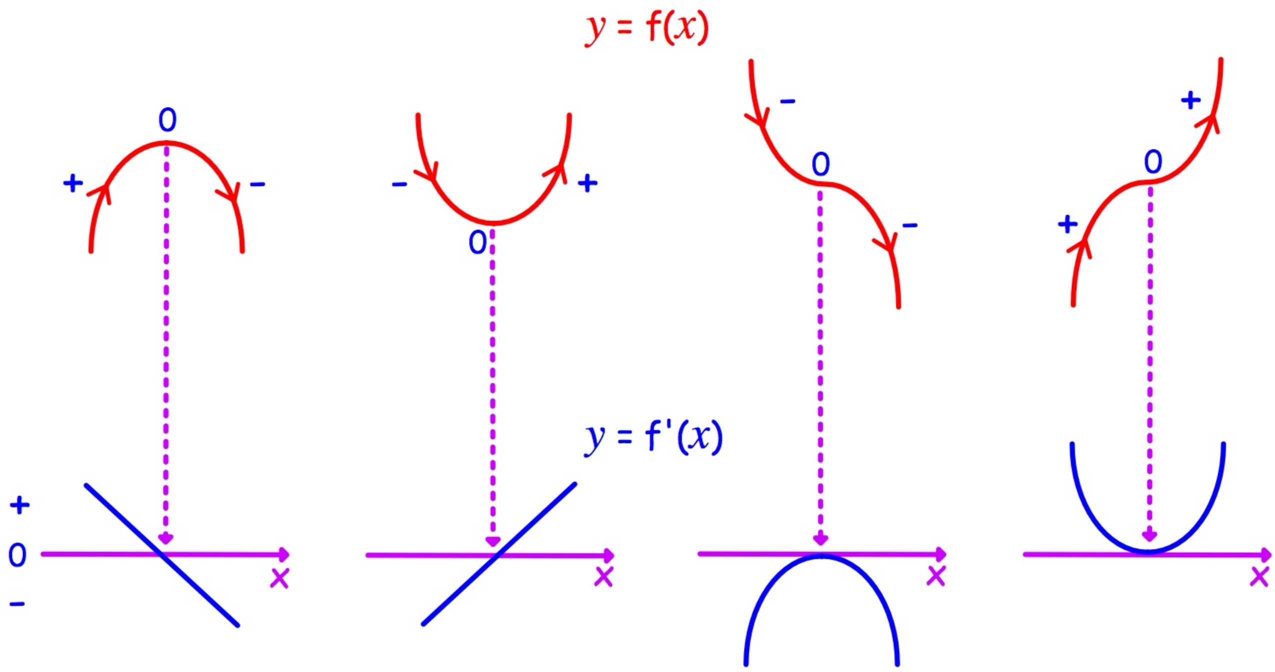
$f'(x) > 0$ $\quad$ $f'(x) < 0$ $\quad$ $f'(x) = 0$

STATIONARY POINTS

Main features of a graph occur at stationary points.

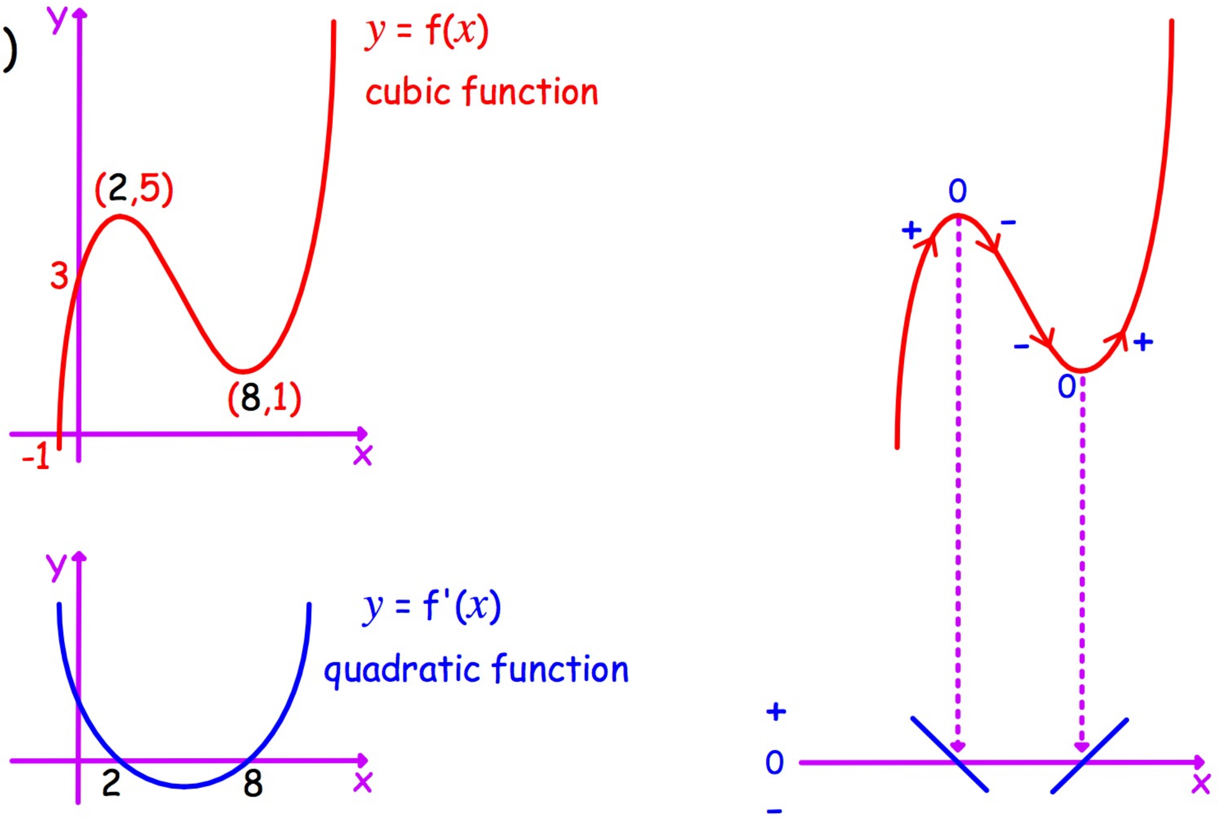
By examining the gradient around stationary points the graph of the gradient can be sketched.

turning points (maximum or minimum) $\quad$ points of inflexion (falling or rising)

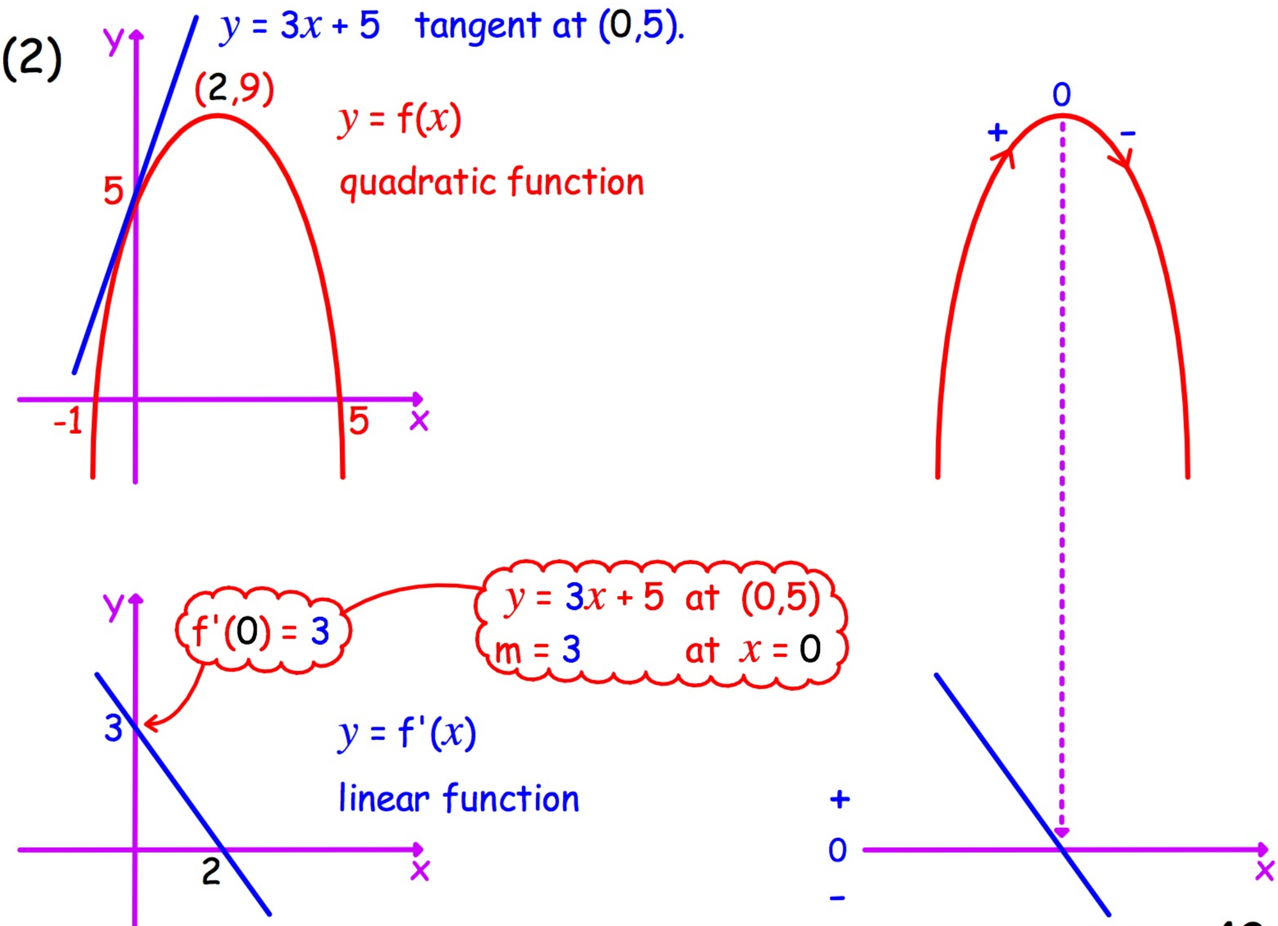


Sketch the graph of $y = f'(x)$

(1)



(2)

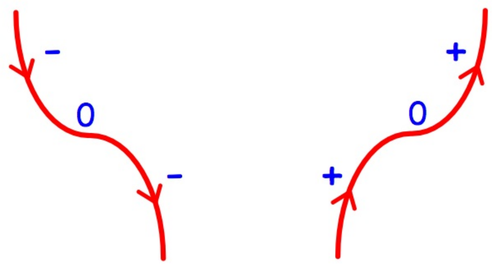


CURVE SKETCHING

The key features of the graph occur at stationary points where $f'(x) = 0$.

turning points
(maximum or minimum)

points of inflexion
(falling or rising)



A NATURE TABLE gives the shape of the graph.

Determine:

(i) the stationary points and their nature

(ii) where the graph meets the axes

(iii) the behaviour of the graph for very large positive and negative values of x.

$y = x^3(x - 4)$

STATIONARY POINTS:

$f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

function stationary where $f'(x) = 0$

$4x^2(x - 3) = 0$

$4x^2 = 0$ or $x - 3 = 0$

$x = 0$ or $x = 3$

$f(x) = x^3(x - 4)$

$f(0) = 0^3(0 - 4) = 0$ point (0,0)

$f(3) = 3^3(3 - 4) = -27$ point (3,-27)

NATURE TABLE:

x	→	0	→	→	3	→
$4x^2$	+	0	+	+	+	+
$x - 3$	-	-	-	-	0	+
$f'(x)$	-	0	-	-	0	+
shape	↘	→	↘	↘	→	↗
nature		point of inflexion (0,0)			minimum TP (3,-27)	

AXES:

y - axis , $x = 0$

$f(0) = 0^3(0 - 4) = 0$

point (0,0)

x - axis , $y = 0$

$x^3(x - 4) = 0$

$x = 0$ or $x = 4$

points (0,0) , (4,0)

BEHAVIOUR FOR VERY LARGE x :

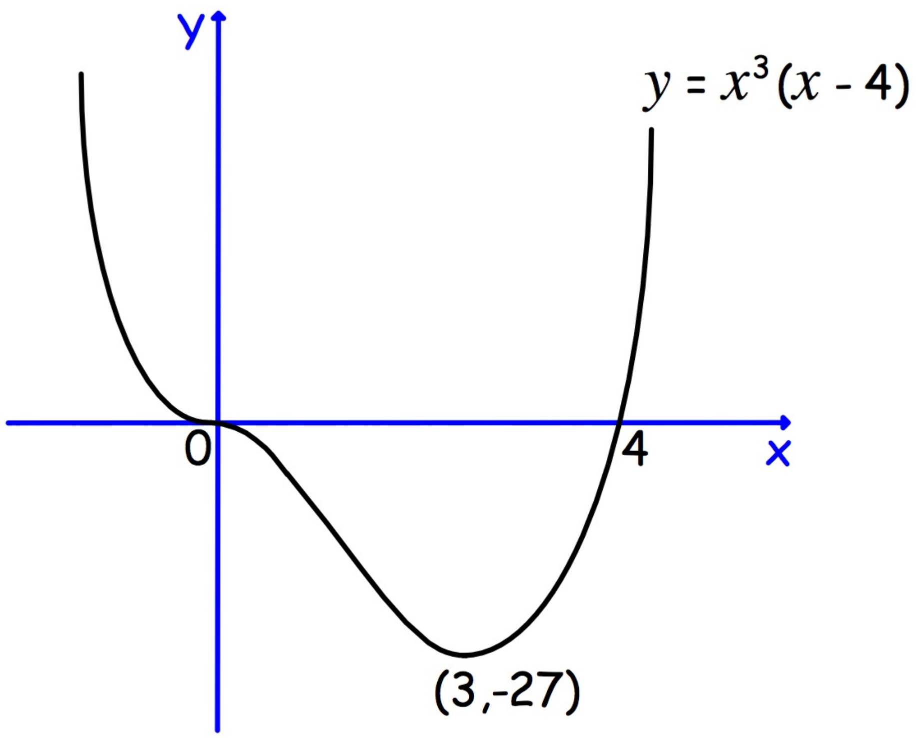
$$y = x^4 - 4x^3$$

as $x \to \infty$ $\quad$ $y \to +x^4$, highest power dominates

as $x \to -\infty$ $\quad$ $y \to +\infty$

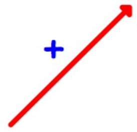
as $x \to +\infty$ $\quad$ $y \to +\infty$

SKETCH:

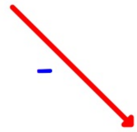


FUNCTION INCREASING AND DECREASING

FUNCTION INCREASING $f'(x) > 0$



FUNCTION DECREASING $f'(x) < 0$



From nature table of $f(x) = x^3(x - 4)$ earlier:

x	→	0	→	→	3	→
$f'(x)$	-	0	-	-	0	+

function increasing where $f'(x) > 0$ $\quad$ $x > 3$

function decreasing where $f'(x) < 0$ $\quad$ $x < 0$ or $0 < x < 3$

Show that the function
$f(x) = x^3 - 6x^2 + 12x - 5$ is never decreasing.

$$f'(x) = 3x^2 - 12x + 12$$
$$= 3(x^2 - 4x + 4)$$
$$= 3(x - 2)^2$$

function decreasing where $f'(x) < 0$

for all values of x $\quad$ $3(x - 2)^2 \geq 0$

so the function is never decreasing.

MAXIMUM and MINIMUM VALUES

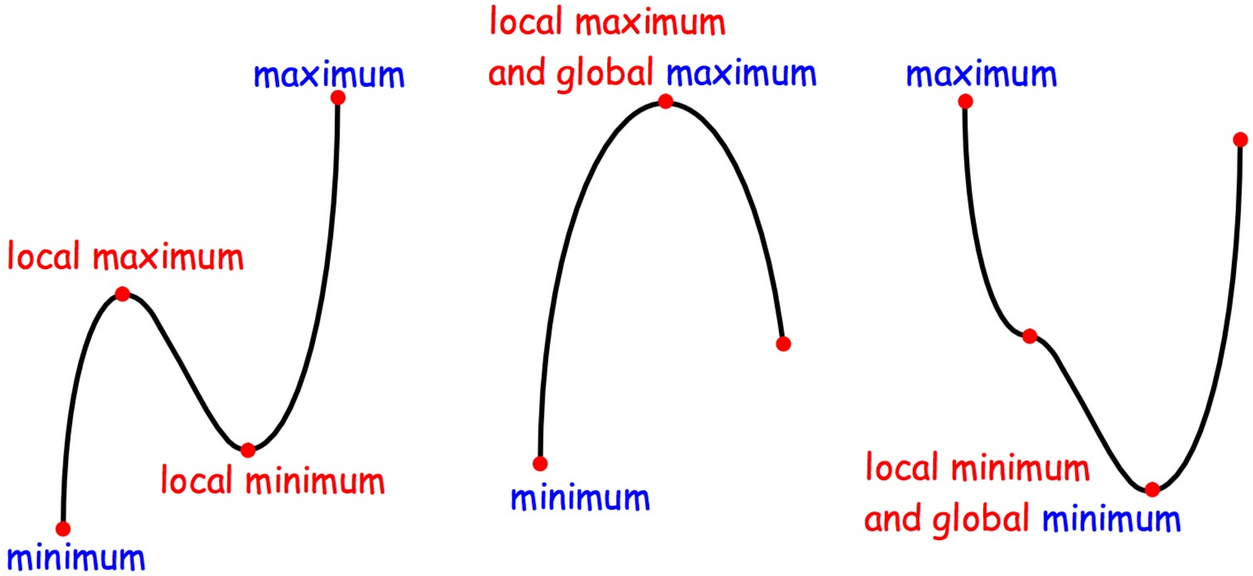
For the maximum and minimum value of a function in a closed interval, examine the values at:

(i) stationary points within the interval

(ii) the end points of the interval

LOCAL max/min values occur at stationary points

GLOBAL max/min values may be at stationary points or at the end points of the interval.



Find the maximum and minimum values of the function
$f(x) = x^2(x^2 - 8)$, $[-1,3]$

$f(x) = x^4 - 8x^2$

$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x + 2)(x - 2)$

function stationary where $f'(x) = 0$

$4x(x + 2)(x - 2) = 0$

$x = 0$ or $x = -2$ or $x = 2$

for the interval $-1 \le x \le 3$, $x = 0$ or $x = 2$

$f(x) = x^4 - 8x^2$

STATIONARY POINTS:

$f(0) = 0^4 - 8 \times 0^2 = 0$

$f(2) = 2^4 - 8 \times 2^2 = -16$

END POINTS OF INTERVAL:

$f(-1) = (-1)^4 - 8(-1)^2 = -7$

$f(3) = 3^4 - 8 \times 3^2 = 9$

maximum value 9 , minimum value -16

OPTIMISATION

Problems involving maxima and minima.

The problem is modelled by a function.
Examine stationary points and end points of interval.

(1) Two numbers have product 16. Find the minimum sum.

product: $xy = 16$

$y = \dfrac{16}{x}$

sum: $x + y$

$= x + \dfrac{16}{x}$

model: $S(x) = x + \dfrac{16}{x}\ ,\ x \neq 0$

Stationary points: $S(x) = x + 16x^{-1}$

$S'(x) = 1 - 16x^{-2} = 1 - \dfrac{16}{x^2}$

function stationary where $S'(x) = 0$

$1 - \dfrac{16}{x^2} = 0$

$x^2 = 16$

$x = \pm 4$

x	→	-4	→	→	4	→
$S'(x)$	+	0	-	-	0	+
shape	↗	—	↘	↘	—	↗
nature		max. value at $x = -4$			min. value at $x = 4$	

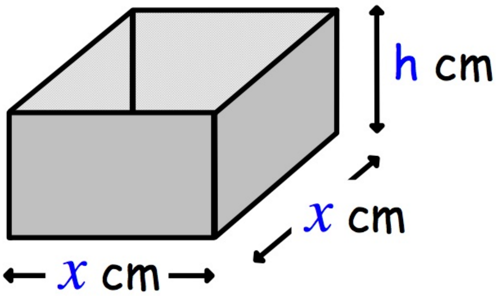
$S(x) = x + \dfrac{16}{x}$

$S(4) = 4 + \dfrac{16}{4} = 8$

minimum sum 8

(2) A square-based open-top cuboid has volume $4000 cm^3$.

Find the dimensions for the minimum surface area.



$$V = lbh$$

$$4000 = x^2 h$$

$$h = \frac{4000}{x^2}$$

area:

$$A = x^2 + 4xh$$

$$= x^2 + 4x \frac{4000}{x^2}$$

$$A(x) = x^2 + \frac{16000}{x} \quad , \; x > 0$$

$$A(x) = x^2 + 1600x^{-1}$$

$$A'(x) = 2x - 1600x^{-2} = 2x - \frac{16000}{x^2}$$

function stationary where $A'(x) = 0$

$$2x - \frac{16000}{x^2} = 0$$

$$2x = \frac{16000}{x^2}$$

$$x^3 = 8000$$

$$x = 20$$

x	→	20	→
$A'(x)$	-	0	+
shape	↘	—	↗
nature	min. value when $x = 20$		

$$h = \frac{4000}{x^2}$$

$$= \frac{4000}{20^2}$$

$$= 10$$

DIMENSIONS: 20 x 20 x 10 cm