FURTHER CALCULUS

DIFFERENTIATE TRIGONOMETRIC FUNCTIONS

In RADIANS only:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Displacement, d metres, of a particle at time t sec.
is given by $d = t^2 + \cos t$.

Find the velocity of the particle after 2 seconds.

$$d(t) = t^2 + \cos t$$

$$d'(t) = 2t - \sin t$$

$$d'(2) = 2 \times 2 - \sin 2$$

calculator set to radians

$$= 3.0907\ldots$$

velocity ≈ 3.1 m/s

CHAIN RULE

The rule to differentiate composite functions.

The order is important. $F(x) = f(g(x))$ — f: acts last

$F'(x) = f'(g(x)) \times g'(x)$ — f': differentiate first

in Leibniz notation: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

(1) $h(x) = (2x - 3)^4$

$h'(x) = 4(2x - 3)^3 \times 2$

$= \underline{\underline{8(2x - 3)^3}}$

(2) $V(r) = \frac{4}{3r + 1}$

$= 4(3r + 1)^{-1}$

$V'(r) = -4(3r + 1)^{-2} \times 3$

$= \underline{\underline{-\frac{12}{(3r + 1)^2}}}$

(3) $w(x) = \sqrt{1 - x^2}$

$= (1 - x^2)^{1/2}$

$w'(x) = {}^1/_2\,(1 - x^2)^{-1/2} \times (-2x)$

$\underline{\underline{-\frac{x}{\sqrt{1 - x^2}}}}$

CHAIN RULE: TRIG. FUNCTIONS

The chain rule gives the results:

In RADIANS only:

$$\frac{d}{dx}\left(\sin(ax+b)\right) = a\cos(ax+b)$$

$$\frac{d}{dx}\left(\cos(ax+b)\right) = -a\sin(ax+b)$$

(1) $h(x) = \sin(2x+3)$

$h'(x) = \underline{\underline{2\cos(2x+3)}}$

(2) $V(t) = \cos 3t$

$V'(t) = \underline{\underline{-3\sin 3t}}$

$$\frac{d}{dx}\left((\sin x)^n\right) = n(\sin x)^{n-1} \times \cos x$$

$$\frac{d}{dx}\left((\cos x)^n\right) = n(\cos x)^{n-1} \times (-\sin x)$$

(1) $h(x) = \sin^3 x$

$= (\sin x)^3$

$h'(x) = 3(\sin x)^2 \times (\cos x)$

$= \underline{\underline{3\sin^2 x \cos x}}$

(2) $f(r) = \sqrt{\cos r}$

$= (\cos r)^{1/2}$

$f'(r) = {}^1/_2 (\cos r)^{-1/2} \times (-\sin r)$

$= \underline{\underline{-\frac{\sin r}{2\sqrt{\cos r}}}}$

A SPECIAL INTEGRAL:

To reverse the chain rule: $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$

NOTE: only for LINEAR FUNCTIONS ie. form $ax + b$

(1) $\int (2x + 3)^3\,dx$

$= \dfrac{(2x+3)^4}{2 \times 4} + C$

$= \frac{1}{8}(2x+3)^4 + C$

(2) $\int \sqrt{(1 - 2u)}\,du$

$= \int (1 - 2u)^{1/2}\,du$

$= \dfrac{(1-2u)^{3/2}}{-2 \times {}^3/_2} + C$

$= -\frac{1}{3}(1 - 2u)^{3/2} + C$

definite integrals:

(3) $\int_0^1 \dfrac{dx}{(x+1)^2}$

$= \int_0^1 (x+1)^{-2}\,dx$

$= \left[\dfrac{(x+1)^{-1}}{1 \times (-1)} \right]_0^1$

$= \left[\dfrac{-1}{x+1} \right]_0^1$

$= \dfrac{-1}{1+1} - \dfrac{-1}{0+1}$

$= -\frac{1}{2} - (-1)$

$= \frac{1}{2}$

SPECIAL INTEGRALS: TRIG. FUNCTIONS

To reverse the chain rule:

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

NOTE: only for LINEAR FUNCTIONS ie. form $ax + b$

(1) $\int (x^2 - 3\cos x)\,dx$

$= \frac{x^3}{3} - 3\sin x + C$

(2) $\int (3 + \sin x)\,dx$

$= 3x + (-\cos x) + C$

$= 3x - \cos x + C$

(3) $\int \cos(2x+3)\,dx$

$= {}^1/_2 \sin(2x+3) + C$

(4) $\int \sin 3u\,du$

$= -{}^1/_3 \cos 3u + C$

(5) $\int \cos(3w - {}^\pi/_4)\,dw$

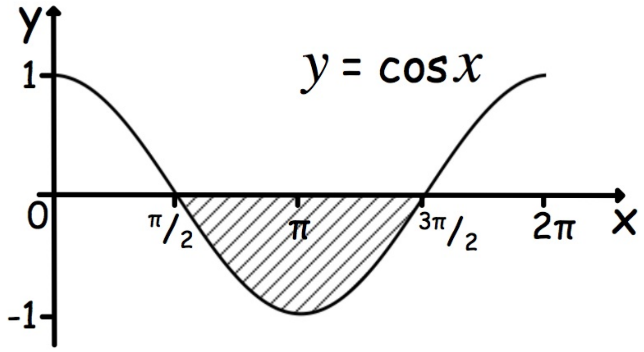
$= {}^1/_3 \sin(3w - {}^\pi/_4) + C$

(6) $\int \sin {}^1/_2 r\,dr$

$= -2\cos {}^1/_2 r + C$

AREAS: TRIG. FUNCTIONS

(1)



$$\int_{\pi/2}^{3\pi/2} \cos x \, dx$$

$$= \left[\sin x \right]_{\pi/2}^{3\pi/2}$$

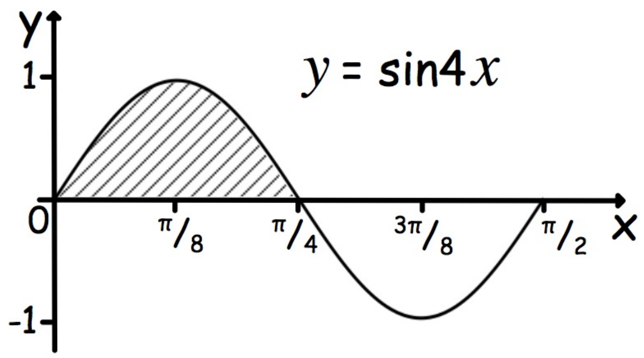
$$= \sin 3\pi/2 - \sin \pi/2$$

$$= -1 - 1$$

$$= -2$$

AREA = 2 units^2

(2)



$$\int_{0}^{\pi/4} \sin 4x \, dx$$

$$= \left[-\tfrac{1}{4} \cos 4x \right]_{0}^{\pi/4}$$

($4 \times \pi/4$) (4×0)

$$= -\tfrac{1}{4} \cos \pi - (-\tfrac{1}{4} \cos 0)$$

$$= -\tfrac{1}{4} \times (-1) - (-\tfrac{1}{4} \times 1)$$

$$= \tfrac{1}{2}$$

AREA = $\tfrac{1}{2}$ units^2

EXPONENTIALS and LOGARITHMS

Any POSITIVE number can be written as a power, a^x.

The logarithm of a number is the index (exponent) to which the base must be raised.

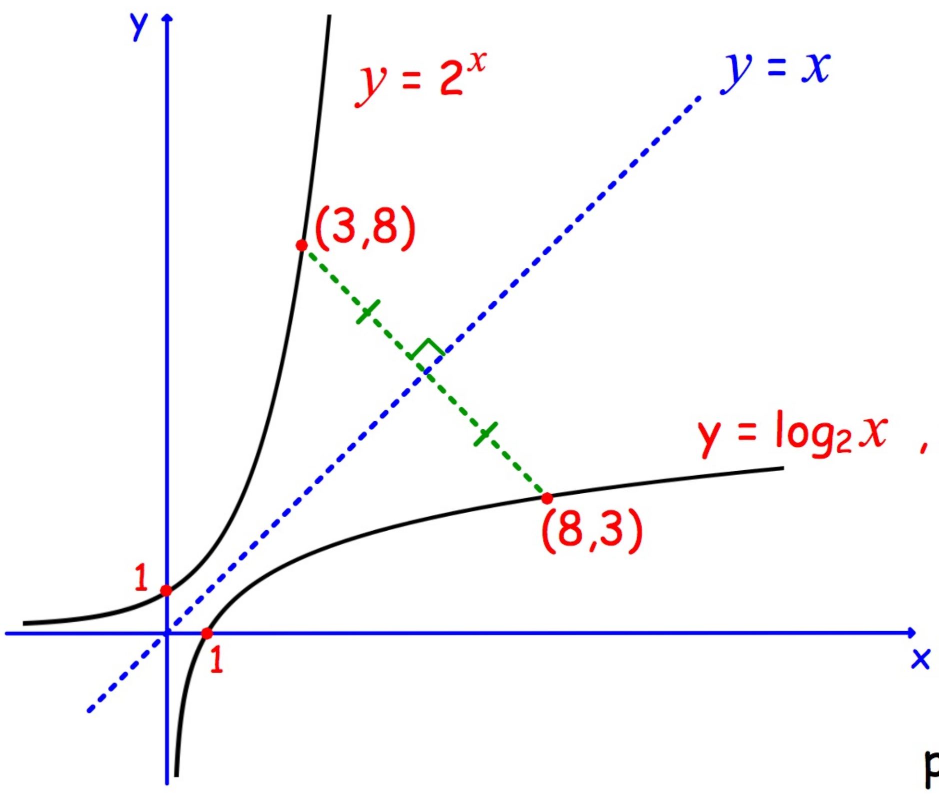
NOTE: can only log. a positive number   ie. $N > 0$

$N = a^x$ INDEX or EXPONENT FORM

$\Leftrightarrow \log_a N = x$ LOGARITHMIC FORM

$49 = 7^2$	$1 = a^0$	$^1/_8 = 2^{-3}$	$27 = 9^{3/2}$
$\log_7 49 = 2$	$\log_a 1 = 0$	$\log_2 {}^1/_8 = -3$	$\log_9 27 = {}^3/_2$

Exponential and logarithmic functions - inverse functions



Simplify

(1) $\log_3 27$

$= \log_3 3^3$

$= \underline{\underline{3}}$

(2) $\log_2 8$

$= \log_2 2^3$

$= \underline{\underline{3}}$

(3) $\log_4 8$

$= \log_4 4^{3/2}$

$= \underline{\underline{3/2}}$

(4) $\log_3 {}^1/_{27}$

$= \log_3 3^{-3}$

$= \underline{\underline{-3}}$

(5) $\log_5 1$

$= \log_5 5^0$

$= \underline{\underline{0}}$

(6) $\log_4 {}^1/_8$

$= \log_4 4^{-3/2}$

$= \underline{\underline{-3/2}}$

Solve

(1) $\log_2 x = 3$

$x = 2^3$

$\underline{\underline{x = 8}}$

(2) $\log_2 x = -3$

$x = 2^{-3}$

$\underline{\underline{x = {}^1/_8}}$

(3) $\log_9 x = -{}^1/_2$

$x = 9^{-1/2}$

$x = \frac{1}{\sqrt{9}}$

$\underline{\underline{x = {}^1/_3}}$

(4) $\log_x 8 = {}^3/_2$

$8 = x^{3/2}$

$8^{2/3} = (x^{3/2})^{2/3}$

$(\sqrt[3]{8})^2 = x^1$

$\underline{\underline{x = 4}}$

LOG RULES:

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a {}^{x}/_{y} = \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

NOTE:

$\log_a 1 = 0$, since $a^0 = 1$

$\log_a a = 1$, since $a^1 = a$

for $0 < N < 1$ $\quad \log_a N < 0$

for $N > 1$ $\quad \log_a N > 0$

eg. $\log {}^{1}/_{2}$ is negative: $\log {}^{1}/_{2} = \log 2^{-1} = -\log 2$

Simplify

$$3 \log_4 2 - \log_4 6 + \log_4 3$$

$$= \log_4 2^3 - \log_4 6 + \log_4 3$$

$$= \log_4 8 - \log_4 6 + \log_4 3$$

$$= \log_4 \left(\frac{8 \times 3}{6} \right)$$

$$= \log_4 4$$

$$= \underline{\underline{1}}$$

Solve

$\log_2 3 + \log_2 x = 3$, $x > 0$

$$\log_2 3x = 3$$

$$3x = 2^3$$

$$\underline{\underline{x = {}^{8}/_{3}}}$$

Solve

$\log(x+2) + \log(x-3) = \log 14 \quad , \; x > 3$

$$\log(x+2)(x-3) = \log 14$$
$$(x+2)(x-3) = 14$$
$$x^2 - x - 6 = 14$$
$$x^2 - x - 20 = 0$$
$$(x+4)(x-5) = 0$$
$$x = -4 \quad \text{or} \quad x = 5$$
$$x > 3, \quad \underline{\underline{x = 5}}$$

Notice the base did not matter.

Using calculator function:

LOG common logarithms, base 10

10^x the corresponding ANTILOG function

Solve

(1) $10^x = 3$

$$x = \log_{10} 3$$
$$= 0.47712\ldots$$
$$\approx \underline{\underline{0.477}}$$

(2) $\log_{10} x = 0.4$

$$x = 10^{0.4}$$
$$= 2.51188\ldots$$
$$\approx \underline{\underline{2.51}}$$

EXPONENTIAL GROWTH and DECAY

DECAY

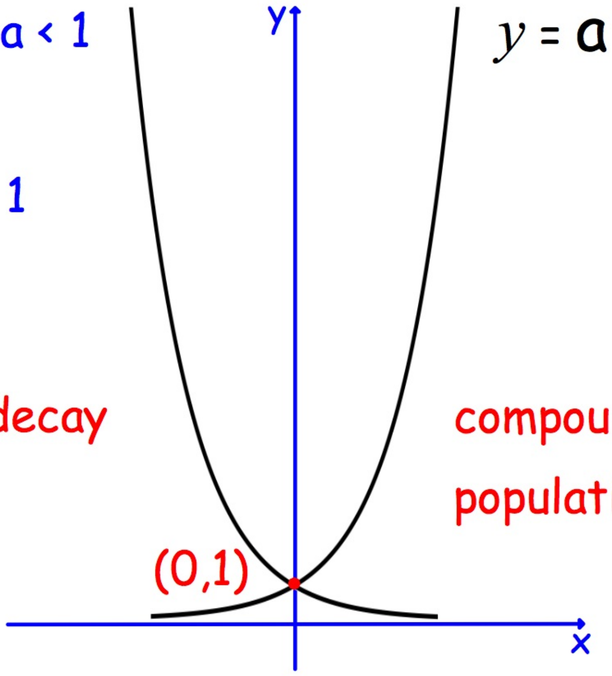
$y = a^x$, $0 < a < 1$

or

$y = a^{-x}$, $a > 1$

GROWTH

$y = a^x$, $a > 1$



radioactive decay
cooling

compound interest
population growths

FORMULAE $A_t = A_0 \times a^t$

$a > 1$	GROWTH
$0 < a < 1$	DECAY

$A_t = A_0 \times a^{kt}$

$k > 0$	GROWTH
$k < 0$	DECAY

initial amount A_0 ,
amount A_t after t iterations

EQUATIONS WITH UNKOWN EXPONONENT:

Log. both sides and use $\log_a x^n = n \log_a x$

$$4^x = 3$$

$$\log_{10} 4^x = \log_{10} 3$$

$$x \log_{10} 4 = \log_{10} 3$$

$$x = \frac{\log_{10} 3}{\log_{10} 4}$$

$$= 0.7924...$$

$$\approx \underline{\underline{0.792}}$$

Money is invested at 10% per year.

How many years for the investment to double ?

$$A_t = A_o (1.10)^t$$

$$200 = 100 (1.10)^t \quad \text{assume } A_o = 100$$

$$(1.10)^t = 2$$

$$\log_{10} (1.10)^t = \log_{10} 2$$

$$t \log_{10} (1.10) = \log_{10} 2$$

$$t = \frac{\log_{10} 2}{\log_{10} (1.10)} = 7.272...$$

<u>8 years</u> required

NATURAL GROWTH and DECAY

Base e ; an irrational number, e = 2.7 1828 1828 4590...

Calculator:

Ln natural logarithms, base e

e^x the corresponding ANTILOG function

The mass m grams of a radioactive isotope after t hours is

$$m_t = m_o e^{-0.02t}$$

Calculate

(a) the mass remaining in an 80 g sample after 10 years.

(b) the time for half the isotope to decay (half-life).

(a)

$$m_t = m_o e^{-0.02t}$$
$$= 80 \times e^{(-0.02 \times 10)}$$
$$= 65.498...$$
$$\approx \underline{\underline{65.5 \text{ grams}}}$$

(b)

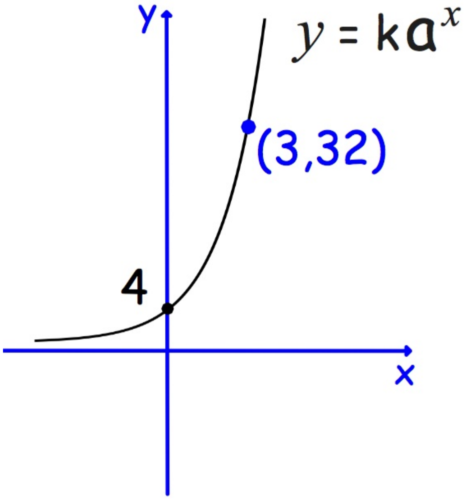
$$m_t = m_o e^{-0.02t}$$
$$50 = 100 e^{-0.02t} \quad \text{assume } m_o = 100$$
$$e^{-0.02t} = 0.5$$

changing from index to log. form

$$-0.02t = \log_e 0.5$$
$$t = \frac{\log_e 0.5}{-0.02} = 34.657... \approx \underline{\underline{34.7 \text{ hours}}}$$

GRAPHS

(1) Find k and a



$y = ka^x$

x y
(0,4) $\quad 4 = k \times a^0$

$4 = k \times 1$

$\underline{\underline{k = 4}}$

$y = 4a^x$

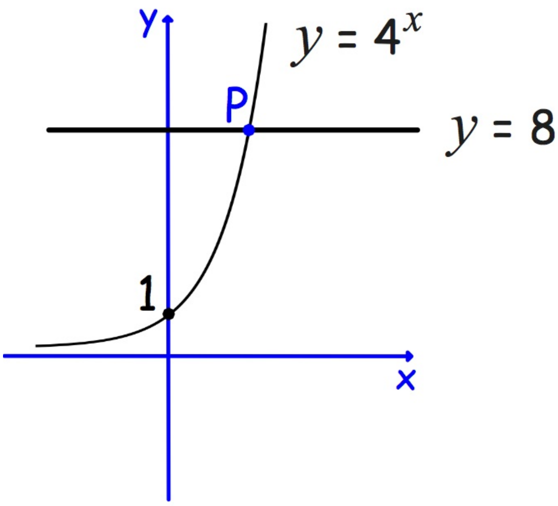
x y
(3,32) $\quad 32 = 4a^3$

$8 = a^3$

$\underline{\underline{a = 2}}$

Equation $y = 4(2)^x$ or $y = 4 \times 2^x$

(2) Find the coordinates of the point of intersection P.



$4^x = 8$

$(2^2)^x = 2^3$

$2^{2x} = 2^3$

$2x = 3$

$x = {}^3/_2$

$\underline{\underline{P\left({}^3/_2 , 8\right)}}$

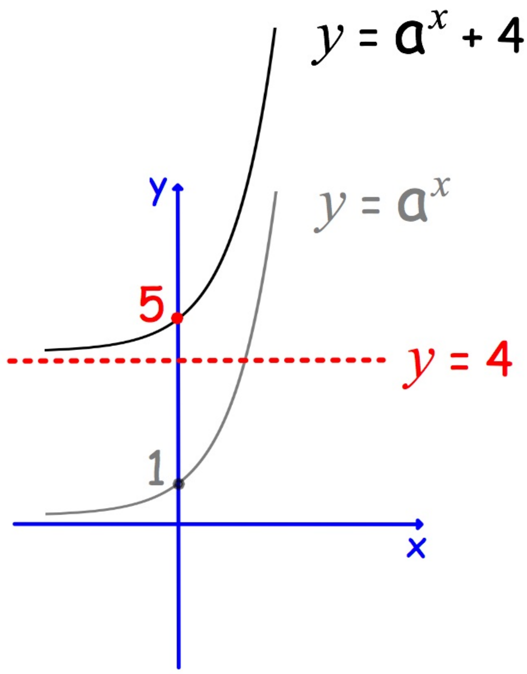
TRANSFORM GRAPHS

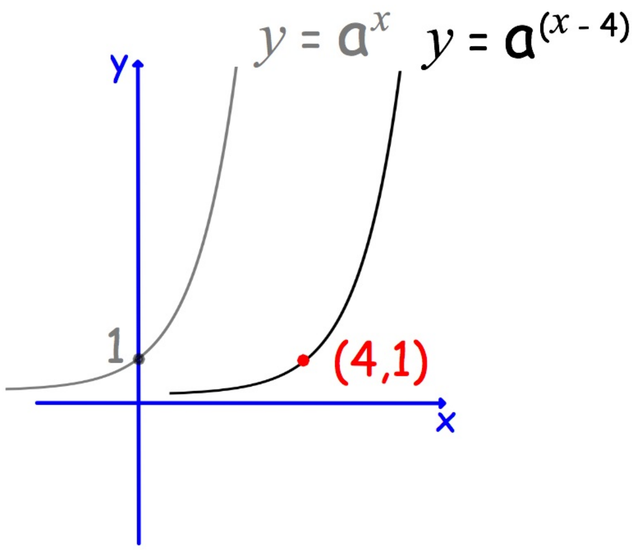
Draw the basic shape of the transformed graph.
Annotate with the images of key points.

	$y = f(x) + k$	$(x, y + k)$
	$y = f(x + k)$	$(x - k, y)$
	$y = kf(x)$	(x, ky)
	$y = f(kx)$	$(^1/_k\, x, y)$
REFLECT in X- axis	$y = -f(x)$	$(x, -y)$
REFLECT in Y- axis	$y = f(-x)$	$(-x, y)$
HALF-TURN about O	$y = -f(-x)$	$(-x, -y)$

(1) $f(x) = a^x + 4$

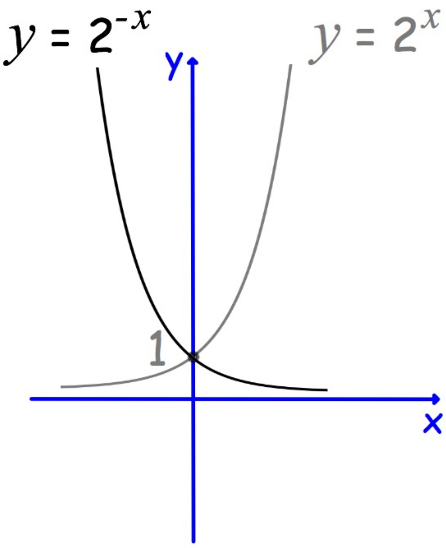
(2) $f(x) = a^{(x - 4)}$



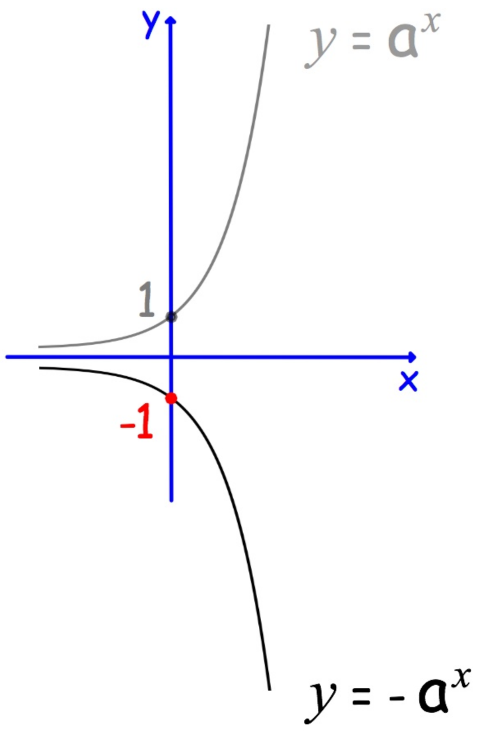


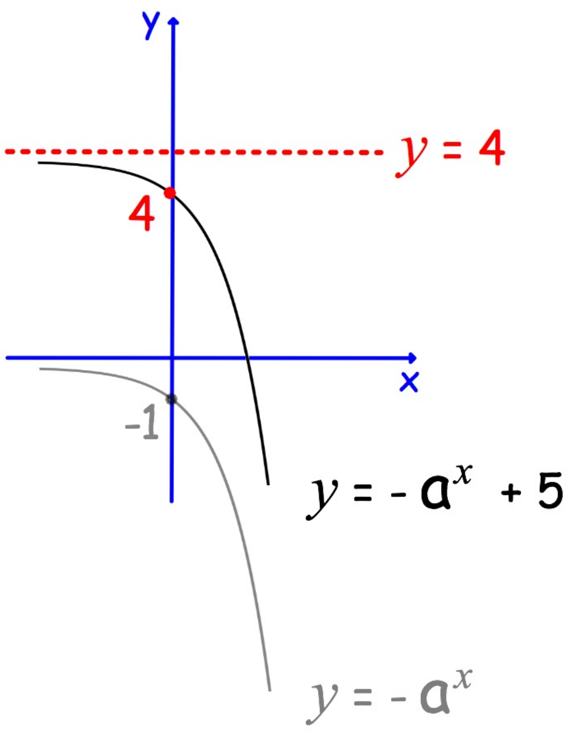
(3) $f(x) = (^1/_2)^x$

$$(^1/_2)^x$$
$$= (2^{-1})^x$$
$$= 2^{-x}$$



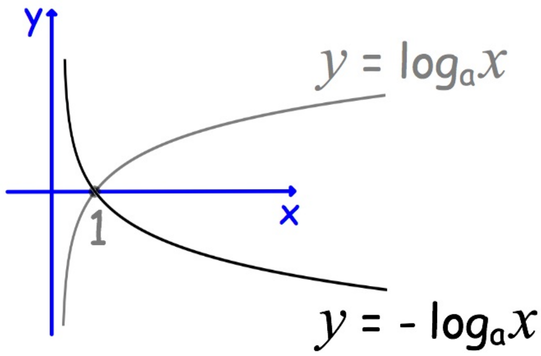
(4) $f(x) = 5 - a^x$



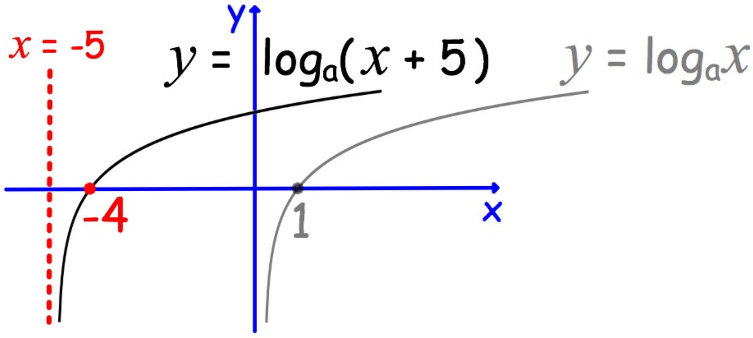


(5) $f(x) = \log_a(^1/x)$

$$\log_a(^1/x)$$
$$= \log_a x^{-1}$$
$$= -\log_a x$$



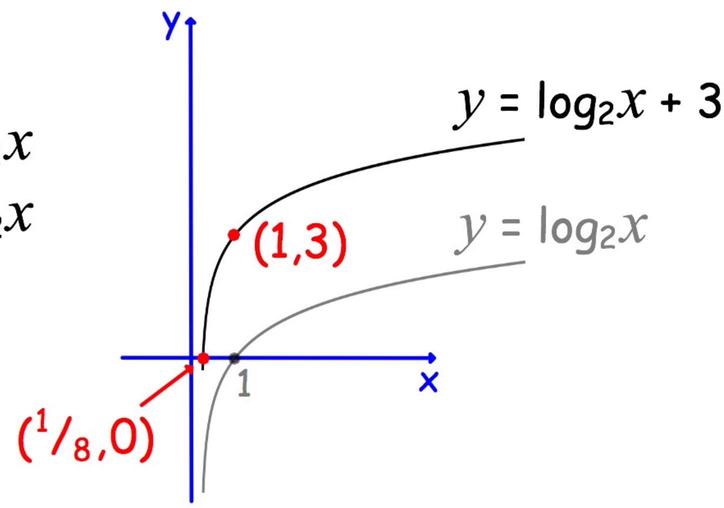
(6) $f(x) = \log_a(x + 5)$



(7) $f(x) = \log_2 8x$

$$\log_2 8x$$
$$= \log_2 8 + \log_2 x$$
$$= 3 + \log_2 x$$

$2^3 = 8$



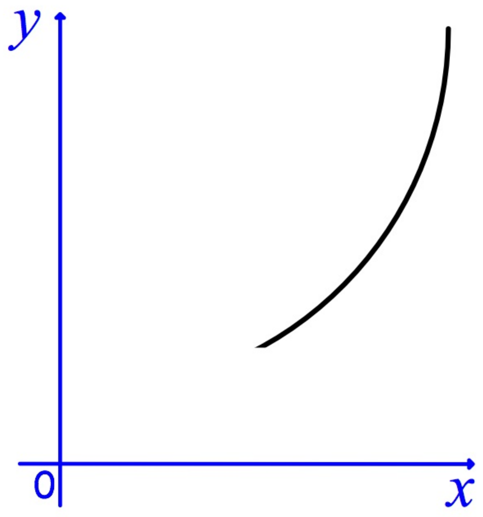
x-axis: $\log_2 8x = 0$

$8x = 2^0$

$8x = 1$

$x = {}^1/_8$

EXPERIMENTAL DATA: FORMULAE



When graphing experimental results exponential and power graphs look similar.

By plotting log. graphs they can be distinguished.

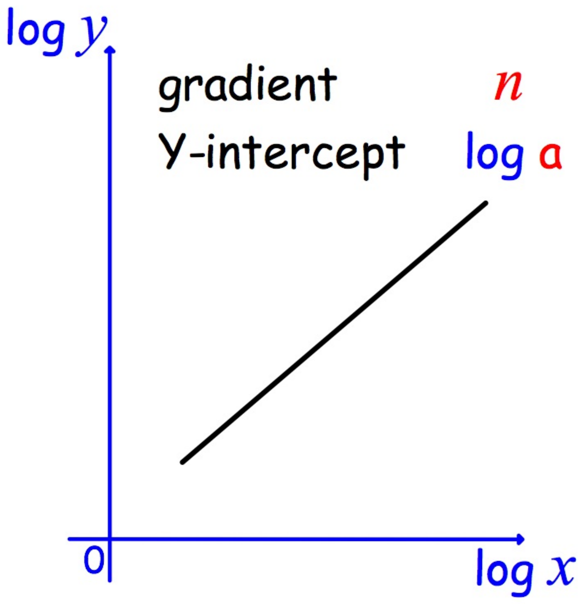
LOG. FORM shows linear relationship $y = mx + C$ and the graph will be a straight line.

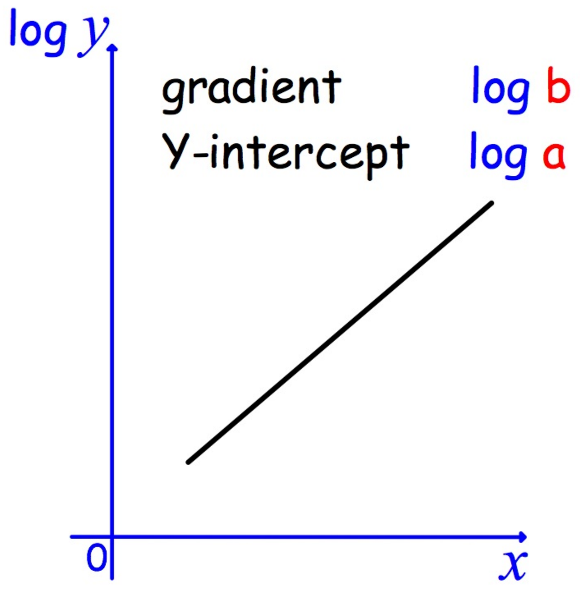
$y = ax^n$

$\log y = n \log x + \log a$

$y = ab^x$

$\log y = (\log b)x + \log a$





Log. rules are used to change from index to log. form.

$$y = ax^n$$

$$\log y = \log (ax^n)$$

$$\log y = \log x^n + \log a$$

$$\log y = n \log x + \log a$$

gradient n

Y-intercept $\log a$

$$y = ab^x$$

$$\log y = \log (ab^x)$$

$$\log y = \log b^x + \log a$$

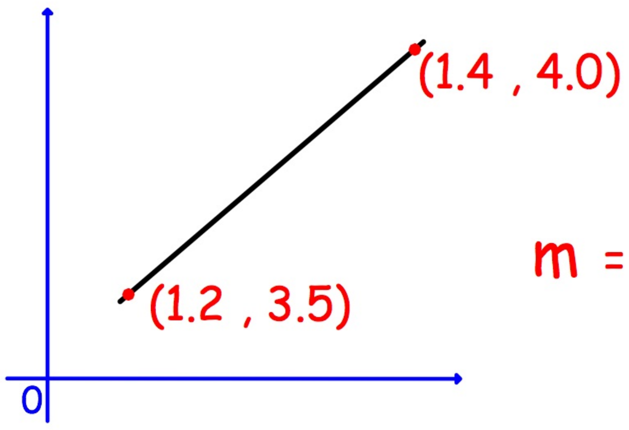
$$\log y = x \log b + \log a$$

$$\log y = (\log b)\, x + \log a$$

gradient $\log b$

Y-intercept $\log a$

EQUATION OF THE LINE



$$m = \frac{4.0 - 3.5}{1.4 - 1.2} = \frac{0.5}{0.2} = 2.5$$

a b
(1.4 , 4.0)

or can use (1.2 , 3.5)

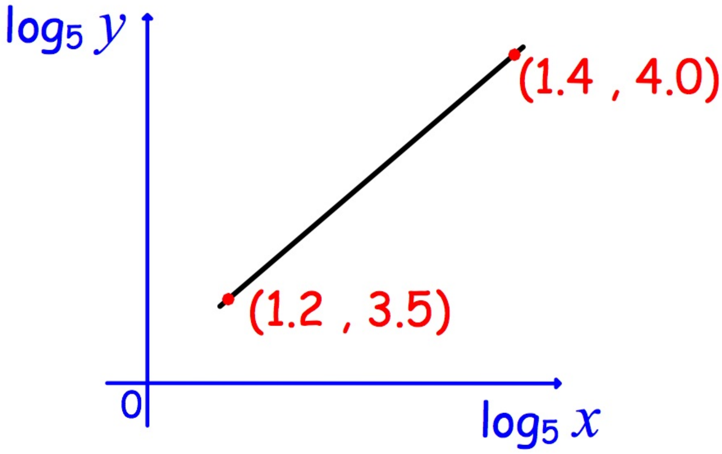
$$y - b = m(x - a)$$

$$y - 4.0 = 2.5\,(x - 1.4)$$

$$y - 4.0 = 2.5\,x - 3.5$$

$$y = 2.5\,x + 0.5$$

(1) Find the formula connecting y and x.



equation of the line $\log y = 2.5 \log x + 0.5$

INDEX FORM $y = ax^n$

LOG. FORM $\log y = n \log x + \log a$

equation of the line $\log y = 2.5 \log x + 0.5$

$n = 2.5$ $\log_5 a = 0.5$

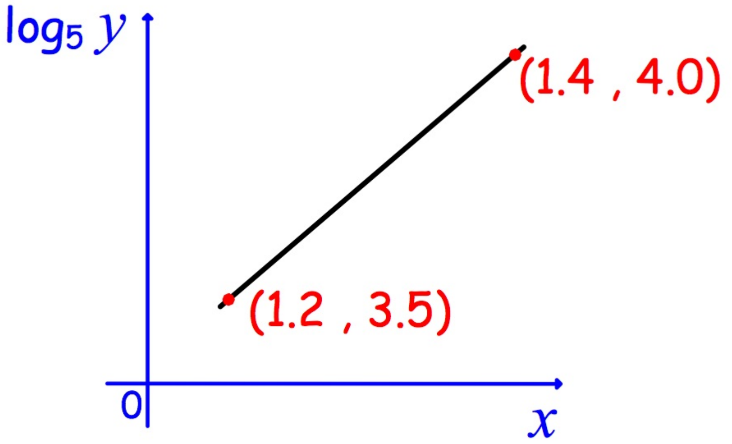
$a = 5^{0.5}$

$a = 2.236...$

$y = ax^n$

$\underline{\underline{y = 2.2x^{2.5}}}$

(2) Find the formula connecting y and x.



equation of the line $\log y = 2.5x + 0.5$

INDEX FORM $y = ab^x$

LOG. FORM $\log y = (\log b)x + \log a$

equation of the line $\log y = 2.5x + 0.5$

$\log_5 b = 2.5$ $\log_5 a = 0.5$

$b = 5^{2.5}$ $a = 5^{0.5}$

$b = 55.901...$ $a = 2.236...$

$y = ab^x$

$y = 2.2(55.9)^x$