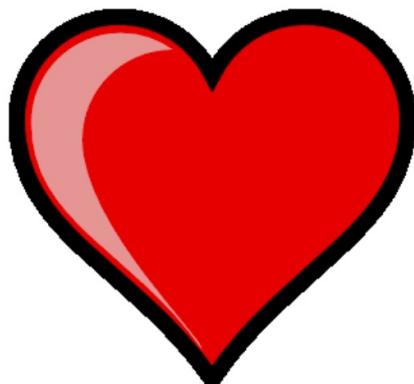


NATIONAL 5 MATHEMATICS COURSE NOTES

Expressions and Formulae

I



MATHS

FORMULAE LIST

The roots of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$

Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Cosine rule: $a^2 = b^2 + c^2 - 2bc \cos A$ or $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

Area of a triangle: $\text{Area} = \frac{1}{2}ab \sin C$

Volume of a sphere: $\text{Volume} = \frac{4}{3}\pi r^3$

Volume of a cone: $\text{Volume} = \frac{1}{3}\pi r^2 h$

Volume of a pyramid: $\text{Volume} = \frac{1}{3}Ah$

Volume of a cylinder: $\text{Volume} = \pi r^2 h$

Standard deviation: $s = \sqrt{\frac{\sum(x - \bar{x})^2}{n-1}} = \sqrt{\frac{\sum x^2 - (\sum x)^2/n}{n-1}}$, where n is the sample size.

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SURDS

Rational numbers can be written as fractions, irrational numbers cannot.
Real numbers are all rational and irrational numbers.

SURDS ARE IRRATIONAL ROOTS:

$\sqrt{2}$, $\sqrt{\frac{5}{9}}$, $\sqrt[3]{16}$ are surds

$\sqrt{25}$, $\sqrt{\frac{4}{9}}$, $\sqrt[3]{-8}$ are not surds as they are 5, $\frac{2}{3}$ and -2 respectively.

RULES:

$$\sqrt{m \times n} = \sqrt{m} \times \sqrt{n}$$

$$\sqrt{\frac{m}{n}} = \frac{\sqrt{m}}{\sqrt{n}}$$

$$(1) \quad \sqrt{72}$$

$$= \sqrt{36} \times \sqrt{2}$$

$$= 6\sqrt{2}$$

$$(2) \quad \sqrt{\frac{5}{9}}$$

$$= \frac{\sqrt{5}}{\sqrt{9}}$$

$$= \frac{\sqrt{5}}{3}$$

largest square number factor of 72

REMOVING BRACKETS:

root \times root

non-root \times non-root

$$2\sqrt{3}(\sqrt{3} - 3)$$

$$= 6 - 6\sqrt{3}$$

$$2\sqrt{3} \times \sqrt{3} = 2 \times \sqrt{9} = 6$$

$$2\sqrt{3} \times 3 = 2 \times 3 \times \sqrt{3} = 6\sqrt{3}$$

ADD/SUBTRACT:

$$\sqrt{36} \times \sqrt{2}$$

$$\sqrt{72} + \sqrt{48} - \sqrt{50}$$

$$= 6\sqrt{2} + 4\sqrt{3} - 5\sqrt{2}$$

$$= \sqrt{2} + 4\sqrt{3}$$

RATIONALISING DENOMINATORS

Remove the surd from the denominator of a fraction:
multiply the 'top' and 'bottom' by the surd.

$$(1) \quad \frac{4}{\sqrt{6}}$$

$$= \frac{4 \times \sqrt{6}}{\sqrt{6} \times \sqrt{6}}$$

$$= \frac{4\sqrt{6}}{6}$$

$$= \frac{2\sqrt{6}}{3}$$

$$(2) \quad \sqrt{\frac{2}{3}}$$

$$= \frac{\sqrt{2}}{\sqrt{3}}$$

$$= \frac{\sqrt{2} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$$

$$= \frac{\sqrt{6}}{3}$$

$$(3) \quad \frac{1}{2\sqrt{3}}$$

$$= \frac{1 \times \sqrt{3}}{2\sqrt{3} \times \sqrt{3}}$$

$$= \frac{\sqrt{3}}{2 \times \sqrt{9}}$$

$$= \frac{\sqrt{3}}{6}$$

To remove a compound surd from the denominator:
multiply the 'top' and 'bottom' by the conjugate surd.

$$\frac{1}{\sqrt{3} + 2}$$

CONJUGATE SURDS:
product rational (no surds)

$$= \frac{1}{(\sqrt{3} + 2)(\sqrt{3} - 2)}$$

$$= \frac{\sqrt{3} - 2}{-1}$$

$$= 3 - 2\sqrt{3} + 2\sqrt{3} - 4$$

$$= -1$$

$$= 2 - \sqrt{3}$$

INDICES

index or exponent
base a^n = $a \times a \times a \times \dots$ to n terms

RULES: same base

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$a^0 = 1$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\frac{2^{-3} \times 2^9}{2^4} = \frac{2^6}{2^4} = 2^2 = 4$$

-3 + 9 = 6 6 - 4 = 2

$$2^{-3} \times 2^3 = 2^0 = 1$$

-3 + 3 = 0

$$(2a^2b)^3 = 2^3 a^6 b^3 = 8a^6 b^3$$

3 × 2 = 6

$$\frac{1}{a^p} = a^{-p}$$

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

Note: $\frac{4}{x^3} = 4x^{-3}$

$$\frac{1}{4x^3} = \frac{1}{4}x^{-3}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

$$8^{\frac{4}{3}} = (\sqrt[3]{8})^4 = 2^4 = 16$$

$$8^{-\frac{4}{3}} = \frac{1}{8^{\frac{4}{3}}} = \frac{1}{16}$$

Brackets:

$$x^3(x^{-3} + x^{-2}) = x^0 + x^1 = 1 + x$$

$-3 + 3 = 0$

$-2 + 3 = 1$

$$2x^3(4x^{-3} + 3x^{-2}) = 8x^0 + 6x^1 = 8 + 6x$$

$$(1 - x^{-1})(1 - x^{-1}) = 1 - x^{-1} - x^{-1} + x^{-2} = 1 - 2x^{-1} + x^{-2}$$

$-1 + -1 = -2$

Surds:

$$(1) \quad 2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}$$

$$\begin{aligned}
 (2) \quad 8^{\frac{3}{2}} &= (\sqrt[2]{8})^3 \\
 &= \sqrt{8} \times \sqrt{8} \times \sqrt{8} \\
 &= 8 \times 2\sqrt{2} \\
 &= 16\sqrt{2}
 \end{aligned}$$

SCIENTIFIC NOTATION (STANDARD FORM)

Used to write very large and very small numbers.

Form $a \times 10^n$

$1 \leq a < 10$ ie. between 1 and 10, excluding 10

n is an INTEGER ie. ...-3,-2,-1,0,1,2,3...

Place the decimal point after the first non-zero digit.
Count the number of places the decimal point moves.

$$257000 = 2.57 \times 10^5$$

2 5 7 0 0 0 •
↑↑↑↑↑↑

negative power - small number between 0 and 1,
the point moves to the right.

$$0.0000257 = 2.57 \times 10^{-5}$$

0.0 0 0 0 2 5 7
↑↑↑↑↑↑

USING THE CALCULATOR

eg. 2.08×10^{-3}

2 . 0 8 $\times 10^n$ (-) 3

Examples: answers in scientific notation
correct to 3 significant figures.

- (1) 1 milligram of hydrogen contains 2.987×10^{20} molecules.
Find the number of molecules in 5 grams.

$$\begin{aligned} 2.987 \times 10^{20} &\times 5000 & 5 \text{ g} &= 5000 \text{ mg} \\ &= 1.4935 \times 10^{24} \\ &= 1.49 \times 10^{24} \text{ molecules} \end{aligned}$$

- (2) The total mass of argon in a flask is 4.15×10^{-2} grams.
The mass of one atom of argon is 6.63×10^{-23} grams.
Find the number of argon atoms in the flask.

$$\begin{aligned} 4.15 \times 10^{-2} &\div 6.63 \times 10^{-23} \\ &= 6.2594\ldots \times 10^{20} \\ &= 6.26 \times 10^{20} \text{ atoms} \end{aligned}$$

ALGEBRAIC EXPRESSIONS

REMOVING BRACKETS

SINGLE BRACKETS: $a \times (b + c) = a \times b + a \times c$

$$(1) \quad 3p(2p + r) \\ = \quad 6p^2 + 3pr$$

$$(2) \quad 2a(3a - b + 5) \\ = \quad 6a^2 - 2ab + 10a$$

Sign changes when multiplying by a negative term:

$$(3) \quad -3(2w - 3y) \\ = \quad -6w + 9y$$

$$(4) \quad -n(4n + 5m) \\ = \quad -4n^2 - 5mn$$

EXPRESSIONS: remove brackets then simplify

no sign change

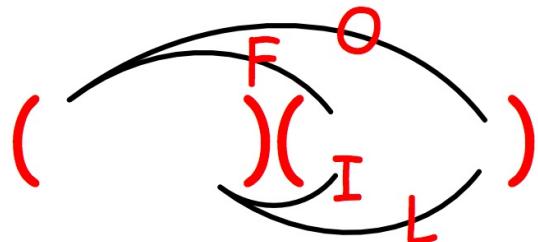
$$(1) \quad 2a + 3a(2 - 3a) \\ = \quad 2a + 6a - 9a^2 \\ = \quad 8a - 9a^2$$

sign change

$$(2) \quad 5 - 3(2a - 3) \\ = \quad 5 - 6a + 9 \\ = \quad 14 - 6a$$

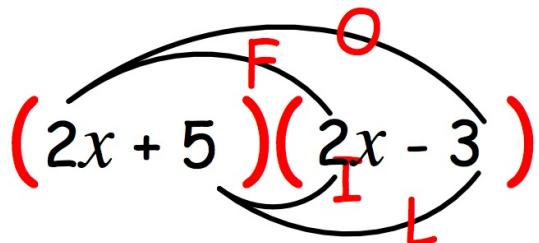
DOUBLE BRACKETS: "FOIL"

Pairs of terms between the two brackets multiplied



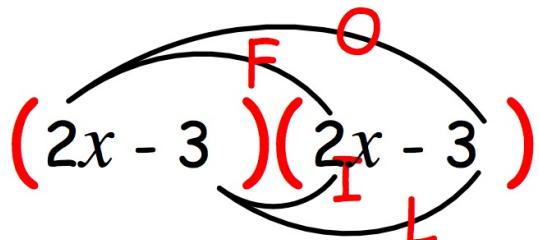
F first pair
O outer pair
I inner pair
L last pair

$$(1) \quad (2x + 5)(2x - 3)$$



$$\begin{aligned} & \text{F} \quad \text{O} \quad \text{I} \quad \text{L} \\ = & 4x^2 - 6x + 10x - 15 \\ = & 4x^2 + 4x - 15 \end{aligned}$$

$$(2) \quad (2x - 3)^2$$



$$\begin{aligned} & \text{F} \quad \text{O} \quad \text{I} \quad \text{L} \\ = & 4x^2 - 6x - 6x + 9 \\ = & 4x^2 - 12x + 9 \end{aligned}$$

Same process for trinomials:

$$\begin{aligned} & (2x - 3)(3x^2 - 2x + 5) \\ = & 2x(3x^2 - 2x + 5) - 3(3x^2 - 2x + 5) \quad \text{sign change} \\ = & 6x^3 - 4x^2 + 10x - 9x^2 + 6x - 15 \\ = & 6x^3 - 4x^2 - 9x^2 + 10x + 6x - 15 \\ = & 6x^3 - 13x^2 + 16x - 15 \end{aligned}$$

EXPRESSIONS: remove brackets then simplify.

$$\begin{aligned} & (x + 2)^2 - (x - 2)^2 \\ = & x^2 + 4x + 4 - (x^2 - 4x + 4) \quad \text{sign change} \\ = & x^2 + 4x + 4 - x^2 + 4x - 4 \\ = & 8x \end{aligned}$$

FACTORIZATION

COMMON FACTORS $a \times b + a \times c = a \times (b + c)$

To FULLY factorise a is the Highest Common Factor.

$$(1) \quad 12w - 18y \\ = 6(2w - 3y)$$

$$(2) \quad 4n^2 + 5mn - n \\ = n(4n + 5m - 1)$$

$$(3) \quad 16p^2 + 8pr \\ = 8p(2p + r)$$

$$(4) \quad 1.4 \times 3.6 - 1.4 \times 1.6 \\ = 1.4 \times (3.6 - 1.6) \\ = 1.4 \times 2 \\ = 2.8$$

DIFFERENCE OF TWO SQUARES

$$a^2 - b^2 = (a + b)(a - b)$$

$$(1) \quad 16 - 9y^2 \\ = 4^2 - (3y)^2 \\ = (4 + 3y)(4 - 3y)$$

$$(2) \quad (6.4)^2 - (3.6)^2 \\ = (6.4 + 3.6)(6.4 - 3.6) \\ = 10 \times 2.8 \\ = 28$$

FULLY factorise:

common factor first

further factorisation

$$(3) \quad 36n^2 - 9 \\ = 9(4n^2 - 1) \\ = 9(2n + 1)(2n - 1)$$

$$(4) \quad n^4 - 16 \\ = (n^2 + 4)(n^2 - 4) \\ = (n^2 + 4)(n + 2)(n - 2)$$

$(6n + 3)(6n - 3)$ is not FULLY factorised

$$\underline{1x^2 + bx + c}$$

a factor pair of c add to b

$$2 + 4 = +6$$

$$2 \times 4 = +8$$

$$(1) \quad x^2 + 6x + 8 = (x + 2)(x + 4)$$

$$-2 + -4 = -6$$

$$-2 \times -4 = +8$$

$$(2) \quad x^2 - 6x + 8 = (x - 2)(x - 4)$$

$$-2 + 4 = +2$$

$$-2 \times 4 = -8$$

$$(3) \quad x^2 + 2x - 8 = (x - 2)(x + 4)$$

$$\underline{ax^2 + bx + c}$$

a factor pair of ac add to b

$$3x \times 1x$$

$$1 \times 8$$

$$2 \times 4$$

$$(1) \quad 3x^2 + 14x + 8 \\ = \underline{(3x + 2)(x + 4)}$$

$$3 \times 8 = 24$$

$$12 \times 2 = 24$$

$$12 + 2 = 14$$

so $12x$ and $2x$

try combinations to make: $12x$ and $2x$

$$\begin{array}{r} 3x \\ \times 1x \\ \hline 1x \end{array} \quad \begin{array}{r} +1 \\ +8 \\ \hline +24x \end{array}$$

$$\begin{array}{r} 3x \\ \times 1x \\ \hline 1x \end{array} \quad \begin{array}{r} +8 \\ +1 \\ \hline +8x \end{array}$$

$$\begin{array}{r} 3x \\ \times 1x \\ \hline 1x \end{array} \quad \begin{array}{r} +2 \\ +4 \\ \hline +2x \end{array}$$

$$\begin{array}{r} 3x \\ \times 1x \\ \hline 1x \end{array} \quad \begin{array}{r} +4 \\ +2 \\ \hline +4x \end{array}$$

correct combination

$$(2) \quad 3x^2 - 14x + 8 \\ = \underline{(3x - 2)(x - 4)}$$

-12 × -2 = +24
3 × 8 = 24
3x² - 14x + 8
-12 + -2 = -14
so -12x and -2x

try combinations to make: -12x and -2x

$$\begin{array}{r} 3x \\ \times 1 \\ \hline 1x \\ \hline -1x \end{array}$$

$$\begin{array}{r} 3x \\ \times -8 \\ \hline -8x \\ \hline -3x \end{array}$$

$$\begin{array}{r} 3x \\ \times -2 \\ \hline 1x \\ \hline -2x \\ \hline -12x \end{array}$$

$$\begin{array}{r} 3x \\ \times -4 \\ \hline 1x \\ \hline -4x \\ \hline -6x \end{array}$$

correct combination

$$(3) \quad 3x^2 + 10x - 8 \\ = \underline{(3x - 2)(x + 4)}$$

+12 × -2 = -24
3 × -8 = -24
3x² + 10x - 8
+12 + -2 = +10
so +12x and -2x

try combinations to make: +12x and -2x

$$\begin{array}{r} 3x \\ \times -1 \\ \hline 1x \\ \hline -1x \end{array}$$

$$\begin{array}{r} 3x \\ \times +8 \\ \hline +8x \\ \hline -3x \end{array}$$

$$\begin{array}{r} 3x \\ \times -2 \\ \hline 1x \\ \hline -2x \\ \hline +12x \end{array}$$

$$\begin{array}{r} 3x \\ \times +4 \\ \hline 1x \\ \hline +4x \\ \hline -6x \end{array}$$

correct combination

ALTERNATIVE

$$3x^2 + 14x + 8$$

12 × 2 = 24
3 × 8 = 24
12 + 2 = 14
so 12x and 2x

$$\begin{aligned}
 (1) \quad & 3x^2 + 14x + 8 \\
 = & 3x^2 + 12x + 2x + 8 && \text{replace } +14x \text{ by } +12x + 2x \\
 = & (3x^2 + 12x) + (2x + 8) && \text{bracket first and last pairs} \\
 = & 3x(x + 4) + 2(x + 4) && \text{fully factorise each bracket} \\
 = & \underline{\underline{(3x + 2)(x + 4)}} && (x + 4) \text{ is a common factor}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & 3x^2 - 14x + 8 \\
 = & 3x^2 - 12x - 2x + 8 \\
 = & 3x(x - 4) - 2(x - 4) \\
 = & \underline{\underline{(3x - 2)(x - 4)}}
 \end{aligned}$$

sign change

$$3x^2 - 14x + 8$$

-12 × -2 = +24
3 × 8 = 24
-12 + -2 = -14
so -12x and -2x

$$\begin{aligned}
 (3) \quad & 3x^2 + 10x - 8 \\
 = & 3x^2 - 2x + 12x - 8 \\
 = & x(3x - 2) + 4(3x - 2) \\
 = & \underline{\underline{(3x - 2)(x + 4)}}
 \end{aligned}$$

-2x first avoids sign change

$$3x^2 + 10x - 8$$

+12 × -2 = -24
3 × -8 = -24
+12 + -2 = +10
so +12x and -2x

COMPLETING THE SQUARE

form $(x + a)^2 + b$

Squaring brackets

$$(x \pm a)^2 = x^2 \pm 2ax + a^2$$

$$(x + 3)^2 = x^2 + 6x + 9$$

rearranging

$$x^2 \pm 2ax = (x \pm a)^2 - a^2$$

$$x^2 + 6x = (x + 3)^2 - 9$$

$$\begin{aligned} (1) \quad & x^2 - 8x + 10 \\ & \quad \text{---} \\ & = (x - 4)^2 - 16 + 10 \\ & = (x - 4)^2 - 6 \end{aligned}$$

$-8 \div 2 = -4$
 $(-4)^2 = 16$

$$\begin{aligned} (2) \quad & x^2 + 3x - 1 \\ & \quad \text{---} \\ & = (x + \frac{3}{2})^2 - \frac{9}{4} - \frac{4}{4} \\ & = (x + \frac{3}{2})^2 - \frac{13}{4} \end{aligned}$$

$3 \div 2 = \frac{3}{2}$
 $(\frac{3}{2})^2 = \frac{9}{4}$

ALGEBRAIC FRACTIONS

SIMPLIFYING:

- (i) fully factorise numerator and denominator
- (ii) cancel common factors

$$(1) \frac{4x^2 - 6x}{4x^2 - 9}$$

$$(2) \frac{2t^2 - 9t + 9}{(t - 3)^3}$$

$$(3) \frac{4x^2y}{4x^2y + 2xy^2}$$

$$= \frac{2x(2x - 3)}{(2x + 3)(2x - 3)}$$

$$= \frac{(2t - 3)(t - 3)}{(t - 3)^2(t - 3)}$$

$$= \frac{2xy \times 2x}{2xy(2x + y)}$$

$$= \frac{2x}{2x + 3}$$

$$= \frac{2t - 3}{(t - 3)^2}$$

$$= \frac{2x}{2x + y}$$

MULTIPLY/DIVIDE: same rules as arithmetic

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

$$(1) \frac{m}{2} \times \frac{6}{m^2}$$

$$(2) \frac{4}{x} \div \frac{y}{2x}$$

$$\div \frac{c}{d} \longrightarrow \times \frac{d}{c}$$

reciprocal

$$= \frac{6m}{2m^2}$$

$$= \frac{4}{x} \times \frac{2x}{y}$$

$$= \frac{2m \times 3}{2m \times m}$$

$$= \frac{8x}{xy}$$

$$= \frac{3}{m}$$

$$= \frac{8}{y}$$

ADD/SUBTRACT: requires a common denominator

$$(1) \quad \frac{m}{2} + \frac{m-1}{3}$$

$$= \frac{3m}{6} + \frac{2(m-1)}{6}$$

$$= \frac{3m + 2m - 2}{6}$$

$$= \frac{5m - 2}{6}$$

$$(2) \quad \frac{y}{x^2} + \frac{1}{2x}$$

$$= \frac{2y}{2x^2} + \frac{1x}{2x^2}$$

$$= \frac{2y + x}{2x^2}$$

$$(3) \quad \frac{3}{t-3} - \frac{3}{t+3}$$

$$= \frac{3(t+3)}{(t-3)(t+3)} - \frac{3(t-3)}{(t-3)(t+3)}$$

$$= \frac{3(t+3) - 3(t-3)}{(t-3)(t+3)}$$

$$= \frac{3t + 9 - 3t + 9}{(t-3)(t+3)}$$

$$= \frac{18}{(t-3)(t+3)}$$

$$(4) \quad \frac{1}{x} + \frac{3}{x(x-3)}$$

$$= \frac{1(x-3)}{x(x-3)} + \frac{3}{x(x-3)}$$

$$= \frac{x-3 + 3}{x(x-3)}$$

$$= \frac{x}{x(x-3)}$$

$$= \frac{1}{x-3}$$