HIGHER MATHEMATICS
COURSE NOTES

UNIT 2

FORMULAE LIST

Circle:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

The equation $(x - a)^2 + (y - b)^2 = r^2$ represents a circle centre (a, b) and radius r.

Scalar Product: $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$, where θ is the angle between **a** and **b**

or $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Trigonometric formulae:

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \sin 2A &= 2\sin A \cos A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}$$

Table of standard derivatives:

$f(x)$	$f'(x)$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$

Table of standard integrals:

$f(x)$	$\int f(x)\,dx$
$\sin ax$	$-\frac{1}{a}\cos ax + C$
$\cos ax$	$\frac{1}{a}\sin ax + C$

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POLYNOMIALS

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots\dots\dots + a_2 x^2 + a_1 x^1 + a_o x^0$$

The COEFFICIENTS a_o , a_1 , a_2 ... a_n are constants, $a_n \neq 0$

The DEGREE is the highest power.

The CONSTANT is the term independent of x. (a_o)

Usually written as descending powers of x.

$(x - 2)(x + 2)(x^3 + 3)$

$= x^5 - 4x^3 + 3x^2 - 12$

degree 5

coefficient of x^5 is 1

coefficient of x^4 is 0

constant is -12

VALUE

$f(x) = 2x^3 + 4x^2 - 10$, find f(-3).

$f(-3) = 2 \times (-3)^3 + 4 \times (-3)^2 - 10 = \underline{\underline{-28}}$

NESTED FORM: uses detached coefficients.

include ALL descending powers of x.

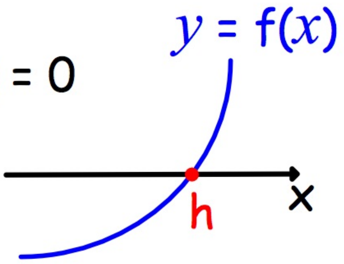
$$2x^3 \quad + 4x^2 \quad + 0x^1 \quad - 10x^0$$

$$\begin{array}{r|rrrrl} -3 & 2 & 4 & 0 & -10 & \\ & & & & & \text{add} \\ & 0 & -6 & 6 & -18 & \\ & \times -3 \nearrow & \times -3 \nearrow & \times -3 \nearrow & & \\ \hline & 2 & -2 & 6 & \underline{\underline{-28}} & \end{array}$$

$\underline{\underline{f(-3) = -28}}$

FACTOR THEOREM:

$$f(h) = 0 \iff x - h \text{ is a factor}$$
$$\iff h \text{ is a root of equation } f(x) = 0$$



(1) Show 3 is a root of $x^3 - 3x^2 - x + 3 = 0$

$$\begin{array}{c|ccccc} 3 & 1 & -3 & -1 & 3 & \\ & & & & & \text{add} \\ & 0 & 3 & 0 & -3 & \\ & \times 3 \nearrow & \times 3 \nearrow & \times 3 \nearrow & & \\ \hline & 1 & 0 & -1 & \underline{\underline{0}} & \end{array}$$

$f(3) = 0$

$\iff \underline{\underline{3 \text{ is a root}}}$

(2) Show $x - 3$ is a factor of $x^3 - 3x^2 - x + 3$

as (1) but conclude $f(3) = 0$

$\iff \underline{\underline{x - 3 \text{ is a factor}}}$

(3) Show $x + 2$ is a factor of $x^3 + 2x^2 - 9x - 18$

$$\begin{array}{c|ccccc} -2 & 1 & 2 & -9 & -18 & \\ & & & & & \text{add} \\ & 0 & -2 & 0 & 18 & \\ & \times -2 \nearrow & \times -2 \nearrow & \times -2 \nearrow & & \\ \hline & 1 & 0 & -9 & \underline{\underline{0}} & \end{array}$$

$f(-2) = 0$

$\iff \underline{\underline{x + 2 \text{ is a factor}}}$

REMAINDER THEOREM:

$$f(x) = (\underbrace{x - h}_{\text{divisor}})\,\underbrace{Q(x)}_{\text{quotient}} + \underbrace{f(h)}_{\text{remainder}}$$

POLYNOMIAL = DIVISOR × QUOTIENT + REMAINDER

$Q(x)$ is a polynomial of degree one lower than $f(x)$

SYNTHETIC DIVISION

When $f(x)$ is divided by $x - h$ the remainder is $f(h)$.

(1) Find the quotient and remainder on dividing
$2x^3 + 5x^2 - 28x - 9$ by $x - 3$

$$\begin{array}{c|ccccccc} 3 & 2 & & 5 & & -28 & & -9 \\ & & & & & & & \text{add} \\ & 0 & & 6 & & 33 & & 15 \\ & & \times 3 \nearrow & & \times 3 \nearrow & & \times 3 \nearrow & \\ \hline & 2 & & 11 & & 5 & & \underline{\underline{6}} \end{array}$$

$f(3) = 6$

$\Longleftrightarrow$ remainder 6

$$2x^3 + 5x^2 - 28x - 9 = (x - 3)(2x^2 + 11x + 5) + 6$$

quotient $2x^2 + 11x + 5$, remainder 6

(2) Find the quotient and remainder on dividing

$4x^3 - 3x + 5$ by $2x + 1$

root $2x + 1 = 0$

$2x = -1$

$x = -^1/_2$

$$\begin{array}{r|rrrrl} -^1/_2 & 4 & 0 & -3 & 5 & \\ & & & & & \text{add} \\ & 0 & -2 & 1 & 1 & \\ & \times -^1/_2 \nearrow & \times -^1/_2 \nearrow & \times -^1/_2 \nearrow & & \\ \hline & 4 & -2 & -2 & \underline{\underline{6}} & \end{array}$$

$f(-^1/_2) = 0$

$\Leftrightarrow x + {}^1/_2$ is a factor

$$\begin{aligned} 4x^3 - 3x + 5 &= (x + {}^1/_2)(4x^2 - 2x - 2) + 6 \\ &= (x + {}^1/_2)\,2(2x^2 - x - 1) + 6 \\ &= (2x + 1)(2x^2 - x - 1) + 6 \end{aligned}$$

$\underline{\underline{\text{quotient } 2x^2 - x - 1 \text{ , remainder } 6}}$

UNKNOWNS

(1) If $2x^3 + x^2 - bx - 1$ divided by $x + 2$ has a remainder of 3 , find the value of b.

$$\begin{array}{r|llll} -2 & 2 & 1 & -b & -1 \\ & & & & \quad\text{add} \\ & 0 & -4 & 6 & 2b-12 \\ & \quad{\times}{-2}\nearrow & \quad{\times}{-2}\nearrow & \quad{\times}{-2}\nearrow & \\ \hline & 2 & -3 & -b+6 & \underline{\underline{2b-13}} \end{array}$$

remainder 3

$\Leftrightarrow f(-2) = 3$

$2b - 13 = 3$

$2b = 16$

$\underline{\underline{b = 8}}$

(2) If $x^3 + ax^2 - 5x - 6$ has a factor $x - 2$, find the value of a and fully factorise.

$$\begin{array}{r|llll} 2 & 1 & a & -5 & -6 \\ & & & & \quad\text{add} \\ & 0 & 2 & 2a+4 & 4a-2 \\ & \quad{\times}2\nearrow & \quad{\times}2\nearrow & \quad{\times}2\nearrow & \\ \hline & 1 & a+2 & 2a-1 & \underline{\underline{4a-8}} \end{array}$$

factor $x - 2$

$\Leftrightarrow f(2) = 0$

$4a - 8 = 0$

$\underline{\underline{a = 2}}$

a = 2

1 4 3

$$x^3 + 2x^2 - 5x - 6 = (x - 2)(1x^2 + 4x + 3)$$

$$= \underline{\underline{(x - 2)(x + 1)(x + 3)}}$$

FACTORISATION

Find a factor h of the CONSTANT such that f(h) = 0

(1) Fully factorise $2x^3 + 5x^2 - 28x - 15$

factors of -15 : ±1 , ±3 , ±5 , ±15

$f(1) = 2\times 1^3 + 5\times 1^2 - 28\times 1 - 15 \neq 0$

$f(-1) = 2\times(-1)^3 + 5\times(-1)^2 - 28\times(-1) - 15 \neq 0$

$f(3) = 2\times 3^3 + 5\times 3^2 - 28\times 3 - 15 = 0$ ✓

3 is a root

3	2	5	-28	-15	
					add
	0	6	33	15	
	×3	×3	×3		
	2	11	5	$\underline{\underline{0}}$	

$f(3) = 0$

$\Longleftrightarrow x - 3$ is a factor

coefficients of a poly. of degree one lower

$$2x^3 + 5x^2 - 28x - 15 = (x - 3)(2x^2 + 11x + 5)$$

$$= \underline{\underline{(x - 3)(2x + 1)(x + 5)}}$$

(2) Given $2x + 1$ is a factor,

fully factorise $2x^3 + 5x^2 - 28x - 15$

and so solve $2x^3 + 5x^2 - 28x - 15 = 0$

root $2x + 1 = 0$

$2x = -1$

$x = -{}^1/_2$

$-{}^1/_2$	2	5	-28	-15	
					add
	0	-1	-2	15	
	$\times -{}^1/_2$	$\times -{}^1/_2$	$\times -{}^1/_2$		
	2	4	-30	$\underline{\underline{0}}$	

$f(-{}^1/_2) = 0$

$\Leftrightarrow x + {}^1/_2$ is a factor

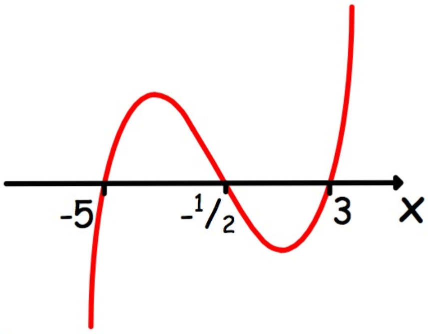
$$\begin{aligned} 2x^3 + 5x^2 - 28x - 15 &= (x + {}^1/_2)(2x^2 + 4x - 30) \\ &= (x + {}^1/_2)\,2(x^2 + 2x - 15) \\ &= (2x + 1)(x^2 + 2x - 15) \\ &= \underline{\underline{(2x + 1)(x + 5)(x - 3)}} \end{aligned}$$

$2x^3 + 5x^2 - 28x - 15 = 0$

$(2x + 1)(x + 5)(x - 3) = 0$

$x = -{}^1/_2$ or $x = -5$ or $x = 3$

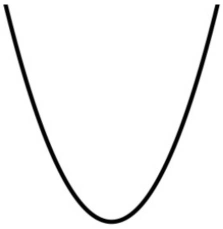
ROOTS are -5, $-{}^1/_2$ and 3



$y = 2x^3 + 5x^2 - 28x - 15$

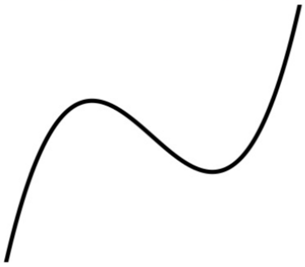
EQUATION FROM THE GRAPH

Quadratics: 'U'



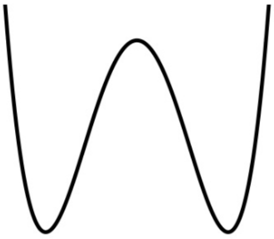
two roots

Cubics: 'S'

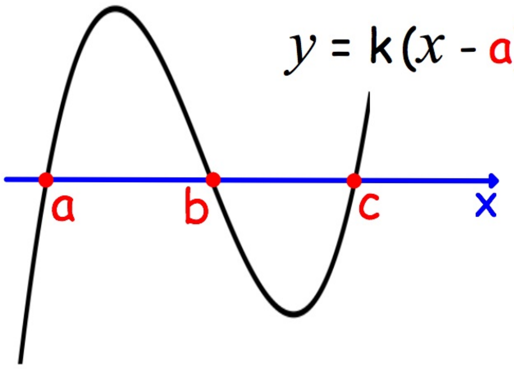


three roots

Quartics: 'W'

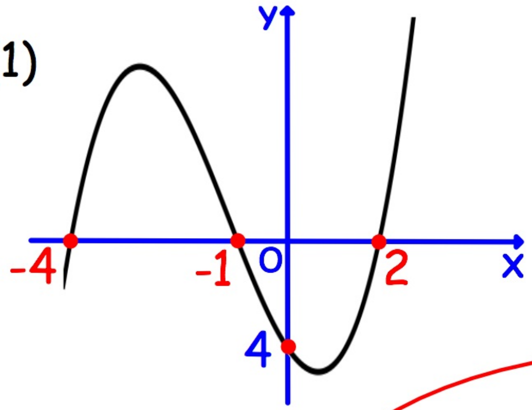


four roots



$y = k(x - a)(x - b)(x - c)$

(1)



for (0,4) substitute $x = 0$ and $y = 4$

$$y = k\,(x - a)(x - b)(x - c)$$

$$y = k\,(x + 4)(x + 1)(x - 2)$$

$$4 = k\,(0 + 4)(0 + 1)(0 - 2)$$

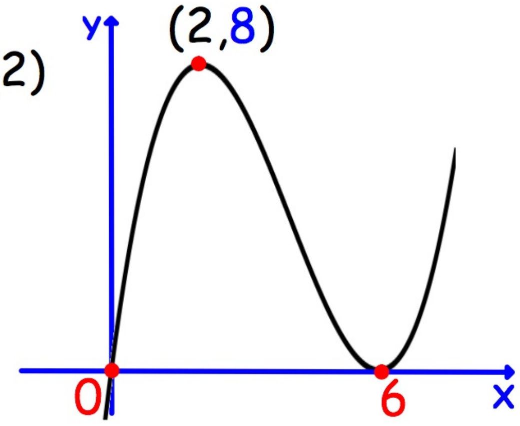
$$4 = -8k$$

$$k = {}^{4}/_{-8}$$

$$k = -\,{}^{1}/_{2}$$

$$\underline{\underline{y = -\,{}^{1}/_{2}\,(x + 4)(x + 1)(x - 2)}}$$

(2)



root $x = 0$

repeated root

$$y = k\,x(x - b)^2$$

$$y = k\,x(x - 6)^2$$

for (2,8) substitute $x = 2$ and $y = 8$

$$8 = k \times 2 \times (2 - 6)^2$$

$$8 = 32k$$

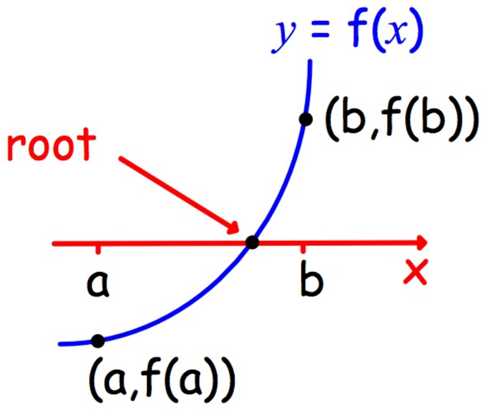
$$k = {}^{8}/_{32}$$

$$k = {}^{1}/_{4}$$

$$\underline{\underline{y = {}^{1}/_{4}\,x(x - 6)^2}}$$

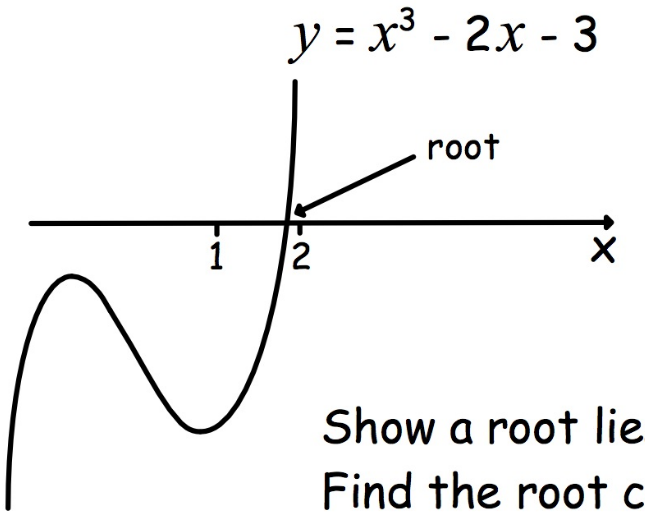
ITERATION

The root is where $f(x) = 0$



To show a root lies between $x = a$ and $x = b$,
show the y-coord. changes sign ie. find f(a) and f(b).

To improve the root repeat the process by using values of x between a and b.



Show a root lies between 1 and 2.
Find the root correct to one decimal place.

Show $x^3 - 2x - 3 = 0$ has a root between 1 and 2.

$f(x) = x^3 - 2x - 3$

$f(1) = 1^3 - 2 \times 1 - 3 = -4$

$f(2) = 2^3 - 2 \times 2 - 3 = +1$

function changed sign, so a root lies between 1 and 2.

consider half-way
between 1 and 2

$f(1.5) = 1.5^3 - 2 \times 1.5 - 3 = -2.625$

sign changes
between 1.5 and 2

$f(1.6) = -2.104$

$f(1.7) = -1.487$

$f(1.8) = -0.768$

$f(1.9) = +0.059$

sign changes
between 1.8 and 1.9

$f(1.85) = -0.368375$

function changed sign, so a root lies between 1.85 and 1.9

$\underline{\underline{\text{ROOT} \approx 1.9}}$

QUADRATIC THEORY

QUADRATIC EQUATIONS

$ax^2 + bx + c = 0$ can be solved by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $b^2 - 4ac$ is 0, 1, 4, 9.... then solve by factorising.

(1) $\dfrac{2}{x} = \dfrac{x + 3}{5}$

$$\frac{2}{x} = \frac{x + 3}{5}$$

$$\begin{aligned} 10 &= x(x + 3) \\ 10 &= x^2 + 3x \\ 0 &= x^2 + 3x - 10 \end{aligned}$$

$$\begin{aligned} x^2 + 3x - 10 &= 0 \\ (x + 5)(x - 2) &= 0 \\ x = -5 \quad \text{or} \quad x &= 2 \end{aligned}$$

(2) Find the roots of $3x^2 - 4x - 9 = 0$

$ax^2 + bx + c = 0$

$a = 3$, $b = -4$, $c = -9$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+4 \pm \sqrt{124}}{6}$$

$$= \frac{4 - \sqrt{124}}{6} \quad \text{or} \quad \frac{4 + \sqrt{124}}{6}$$

$$= -1.189... \quad \text{or} \quad 2.522...$$

$$x = -1.2 \quad \text{or} \quad x = 2.5$$

$$\begin{aligned} &b^2 - 4ac \\ &= (-4)^2 - 4 \times 3 \times (-9) \\ &= 124 \end{aligned}$$

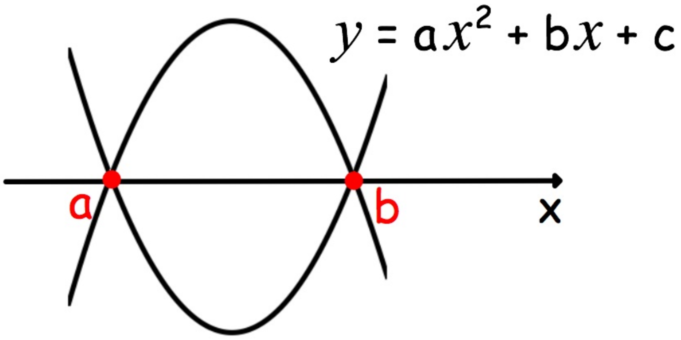
QUADRATIC INEQUALITIES

$(x - a)(x - b) > 0$ or similar

solution is one of two posibilities: (i) $a < x < b$

(ii) $x < a$ or $x > b$

SKETCH GRAPH: $a > 0$ ∪ ie. positive $a < 0$ ∩ ie. negative



$ax^2 + bx + c > 0$

curve above axis

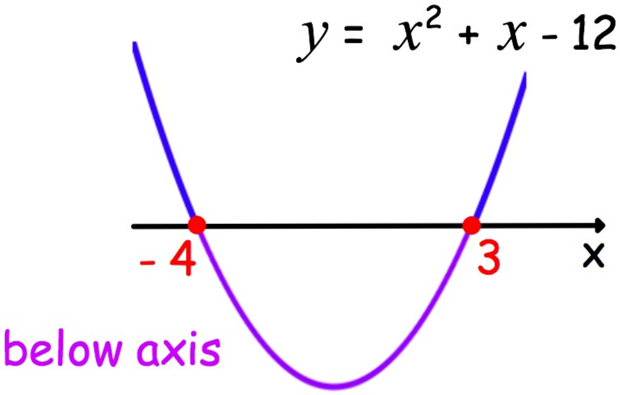
$ax^2 + bx + c < 0$

curve below axis

(1) $x^2 + x - 12 < 0$

$(x + 4)(x - 3) < 0$

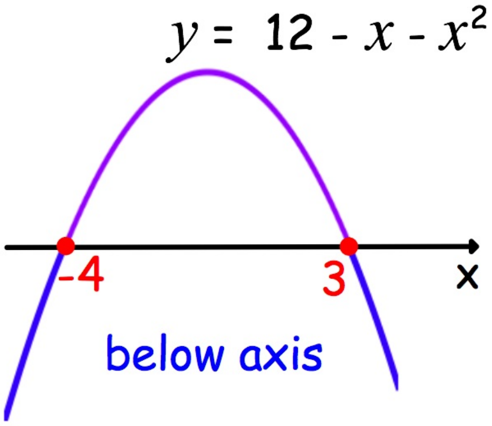
$\underline{\underline{-4 < x < 3}}$



(2) $12 - x - x^2 \leq 0$

$(4 + x)(3 - x) \leq 0$

$\underline{\underline{x \leq -4 \text{ or } x \geq 3}}$



DISCRIMINANT

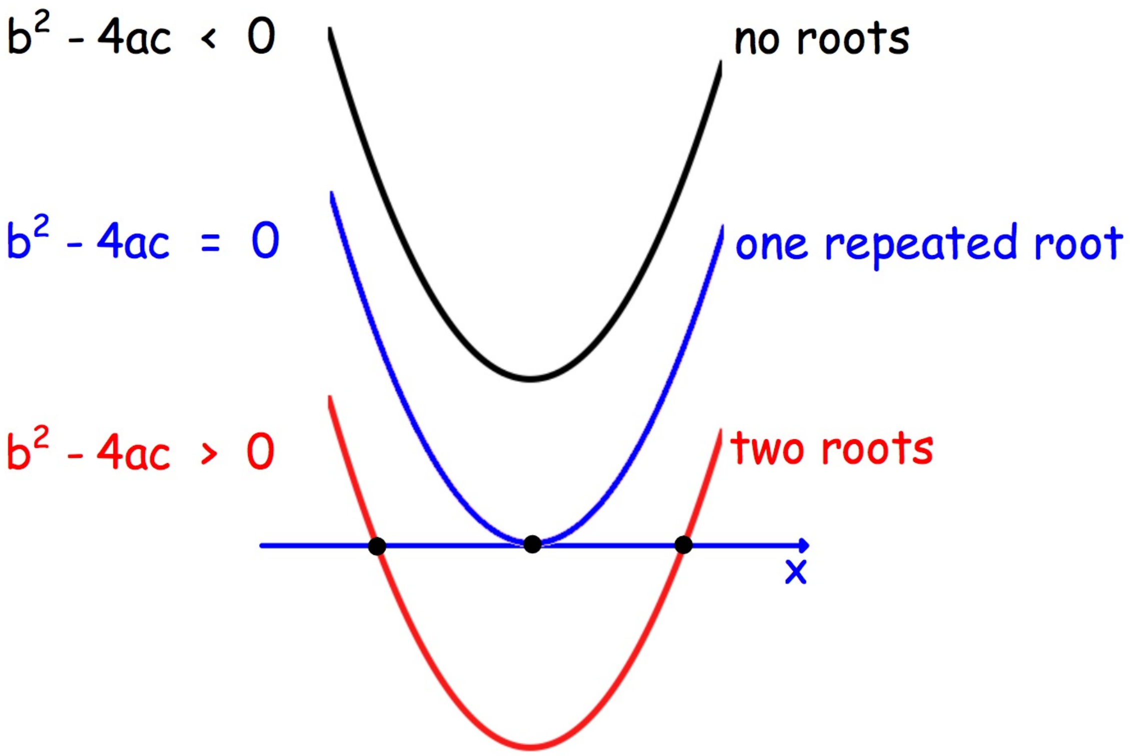
The quadratic equation $ax^2 + bx + c = 0$

can be rearranged to $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula finds the ROOTS of the equation.

$\pm\sqrt{\text{positive}}$	two solutions
$\pm\sqrt{\quad 0 \quad}$	one solution
$\pm\sqrt{\text{negative}}$	no solution

The DISCRIMINANT $b^2 - 4ac$
is used to determine the NATURE of the roots.



NATURE OF THE ROOTS

$b^2 - 4ac > 0$ TWO REAL AND DISTINCT ROOTS

$b^2 - 4ac = 0$ TWO REAL AND EQUAL ROOTS

$b^2 - 4ac < 0$ NO REAL ROOTS

NOTE:

(i) condition for REAL ROOTS $b^2 - 4ac \geq 0$

(ii) if $b^2 - 4ac$ is a square number 0, 1, 4, 9.... then the roots are RATIONAL, otherwise IRRATIONAL. (SURD)

(1) Find the nature of the roots of $3x^2 - 4x - 9 = 0$

$a = 3$, $b = -4$, $c = -9$

$b^2 - 4ac = (-4)^2 - 4 \times 3 \times (-9) = 124$

$b^2 - 4ac > 0 \Rightarrow \underline{\underline{\text{two real and distinct roots}}}$

(and irrational)

(2) Find the nature of the roots of $2x^2 - x + 1 = 0$

$a = 2$, $b = -1$, $c = 1$

$b^2 - 4ac = (-1)^2 - 4 \times 2 \times 1 = -7$

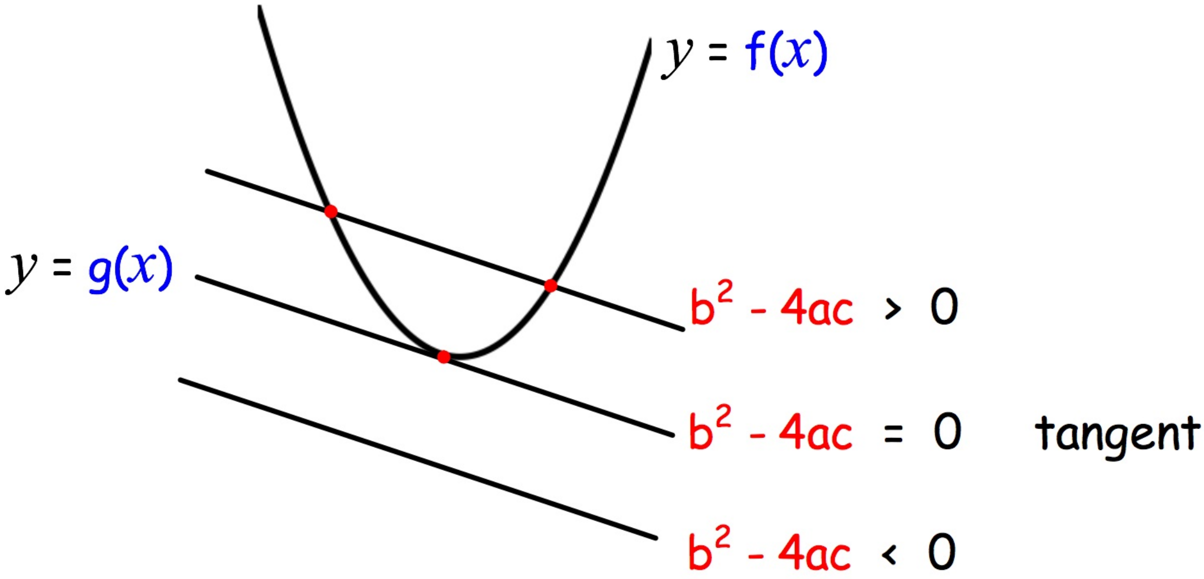
$b^2 - 4ac < 0 \Rightarrow \underline{\underline{\text{no real roots}}}$

INTERSECTION OF A LINE AND CURVE

Substitution $f(x) = g(x)$

results in $ax^2 + bx + c = 0$

Discriminant $b^2 - 4ac$ distinguishes between:



TANGENCY

A tangent is a line that touches the curve at one point.

The substitution results in a quadratic equation
with one solution:

$$b^2 - 4ac = 0 \Rightarrow \text{EQUAL ROOTS}$$

so line is a tangent

Show that the line $y = 5x - 2$ is a tangent to the curve $y = 2x^2 + x$ and find the point of contact.

$$2x^2 + x = 5x - 2$$

$$2x^2 - 4x + 2 = 0$$

$$1x^2 - 2x + 1 = 0$$

$$b^2 - 4ac = (-2)^2 - 4 \times 1 \times 1 = 0$$

$b^2 - 4ac = 0 \Rightarrow$ line is a tangent

$$(x - 1)^2 = 0$$

$$x = 1$$

$$\begin{aligned} y &= 5x - 2 \\ &= 5 \times 1 - 2 \\ &= 3 \end{aligned}$$

point of contact (1,3)

alternative:

$$2x^2 + x = 5x - 2$$

$$2x^2 - 4x + 2 = 0$$

$$1x^2 - 2x + 1 = 0$$

$$(x - 1)^2 = 0$$

$$x = 1$$

one point of contact

$\Rightarrow$ line is a tangent

$$\begin{aligned} y &= 5x - 2 \\ &= 5 \times 1 - 2 \\ &= 3 \end{aligned}$$

point of contact (1,3)

UNKNOWNS

(1) Given $6x^2 + 12x + k = 0$ has REAL roots, find k.

$a = 6 \ , \ b = 12 \ , \ c = k$

$b^2 - 4ac = 12^2 - 4 \times 6 \times k = 144 - 24k$

for real roots ⇒ $b^2 - 4ac \geq 0$

$$144 - 24k \geq 0$$
$$-24k \geq -144$$
$$\underline{\underline{k \leq 6}}$$

(2) Show that $(2k + 4)x^2 + (3k + 2)x + (k - 2) = 0$ always has REAL roots.

$a = (2k + 4) \ , \ b = (3k + 2) \ , \ c = (k - 2)$

$$\begin{aligned} b^2 - 4ac &= (3k + 2)^2 - 4(2k + 4)(k - 2) \\ &= (9k^2 + 12k + 4) - 4(2k^2 - 8) \\ &= 9k^2 + 12k + 4 - 8k^2 + 32 \\ &= k^2 + 12k + 36 \\ &= (k + 6)^2 \end{aligned}$$

for all values of k , $(k + 6)^2 \geq 0$

$b^2 - 4ac \geq 0$ ⇒ real roots

<u>so roots are always real</u>

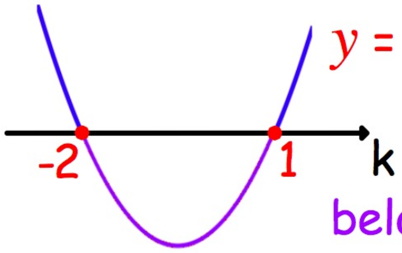
(3) Find the values of k for which the equation $x^2 - 2kx + 2 - k = 0$ has no real roots.

$a = 1$, $b = (-2k)$, $c = (2 - k)$

$$b^2 - 4ac = (-2k)^2 - 4 \times 1 \times (2 - k)$$
$$= 4k^2 - 8 + 4k$$
$$= 4k^2 + 4k - 8$$

no real roots

$$\Rightarrow b^2 - 4ac < 0$$
$$4k^2 + 4k - 8 < 0$$
$$k^2 + k - 2 < 0$$
$$(k + 2)(k - 1) < 0$$
$$\underline{\underline{-2 < k < 1}}$$



$y = (k + 2)(k - 1)$

below axis for $(k + 2)(k - 1) < 0$

(4) If the line gradient 3 , $y = 3x + C$, is a tangent to the curve $y = x^2 + 1$, find C.

$$x^2 + 1 = 3x + C$$
$$1x^2 - 3x + 1 - C = 0$$

$a = 1$, $b = -3$, $c = (1 - C)$

$$b^2 - 4ac = (-3)^2 - 4 \times 1 \times (1 - C)$$
$$= 9 - 4 + 4C$$
$$= 4C + 5$$

line a tangent

$$\Rightarrow b^2 - 4ac = 0$$
$$4C + 5 = 0$$
$$4C = -5$$
$$\underline{\underline{C = -{}^5/_4}}$$