INTEGRAL CALCULUS

ANTI-DIFFERENTIATION

Integration reverses the effect of differentiation.

RULE: $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C \quad , \; n \neq -1$

ie. not for $\int \frac{1}{x} \, dx$

OTHER RULES:

$$\int [\, f(x) + g(x) \,] \, dx = \int f(x) \, dx + \int g(x) \, dx$$

$$\int k \, f(x) \, dx = k \int f(x) \, dx$$

TERMINOLOGY:

The INTEGRAL of x^n with respect to x

The INTEGRAND is x^n

The CONSTANT OF INTEGRATION is C

INDEFINITE INTEGRAL

DIFFERENTIAL EQUATIONS

Contain a derivative $\frac{dy}{dx}$. Solve by integrating.

(1) Solve the equation $\frac{dy}{dx} = 3x^2$
if $y = 3$ when $x = 1$

$$y = \int 3x^2 \, dx$$

GENERAL SOLUTION $y = x^3 + C$

$y = 3$ when $x = 1$

$$3 = 1^3 + C$$
$$C = 2$$

PARTICULAR SOLUTION $\underline{\underline{y = x^3 + 2}}$

(2) The gradient of a curve is given by $f'(x) = x^2 - 5$.
It passes through the point (3,- 4).

Find the equation of the curve.

$$f(x) = \int (x^2 - 5) \, dx$$
$$= \frac{x^3}{3} - 5x + C$$

$f(3) = -4$

$$f(3) = \frac{3^3}{3} - 5 \times 3 + C$$
$$-4 = 9 - 15 + C$$
$$C = 2$$

$$\underline{\underline{y = {}^1/_3 \, x^3 - 5x + 2}}$$

INTEGRATION

(1) $\int (x^4 + x - 2)\, dx$

$= \frac{x^5}{5} + \frac{x^2}{2} - 2x + C$

$= \frac{1}{5}x^5 + \frac{1}{2}x^2 - 2x + C$

(2) $\int 6x^3\, dx$

$= \frac{6x^4}{4} + C$

$= \frac{3}{2}x^4 + C$

(3) $\int (u^{-2} + 2u - 1)\, du$

$= \frac{u^{-1}}{-1} + \frac{2u^2}{2} - u + C$

$= -u^{-1} + u^2 - u + C$

(4) $\int 6x^{-3}\, dx$

$= \frac{6x^{-2}}{-2} + C$

$= -3x^{-2} + C$

INDICES $\frac{1}{a^p} = a^{-p}$ $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

Use indices rules to express terms in the form x^n.

(1) $\int \frac{1}{3x^2}\, dx$ or $\int \frac{dx}{3x^2}$

$= \frac{1}{3} \int x^{-2}\, dx$

$= \frac{1}{3} \times \frac{x^{-1}}{-1} + C$

$= -\frac{1}{3x} + C$

(2) $\int \sqrt{x^3}\, dx$

$= \int x^{\frac{3}{2}}\, dx$

$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$

$= \frac{2}{5}x^{\frac{5}{2}} + C$

BRACKETS AND QUOTIENTS

Integrate sums and differences of terms x^n,
so 'break' brackets and 'split' quotients.

(1) $\int (2x - 3)^2 \, dx$

$$= \int (4x^2 - 12x + 9) \, dx$$

$$= \frac{4x^3}{3} - \frac{12x^2}{2} + 9x + C$$

$$= \frac{4}{3}x^3 - 6x^2 + 9x + C$$

(2) $\int \frac{x - 1}{\sqrt{x}} \, dx$

$$\frac{x^1}{x^{1/2}} - \frac{1}{x^{1/2}}$$

$$= x^{1/2} - x^{-1/2}$$

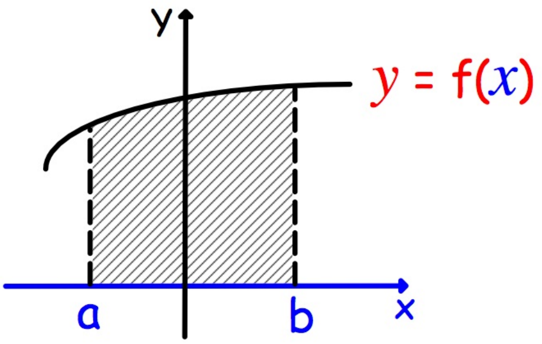
$$= \int (x^{1/2} - x^{-1/2}) \, dx$$

$$= \frac{2}{3}x^{3/2} - \frac{2}{1}x^{1/2} + C$$

$$= \frac{2}{3}x^{3/2} - 2x^{1/2} + C$$

AREA

The area between the curve and the x-axis is given by the DEFINITE INTEGRAL:



$$\text{AREA} = \int_a^b f(x)\,dx$$

lower limit is a
upper limit is b

EVALUATING DEFINITE INTEGRALS

if $f(x) = F'(x)$, $\int_a^b f(x)\,dx = F(b) - F(a)$

written $\left[F(x)\right]_a^b$

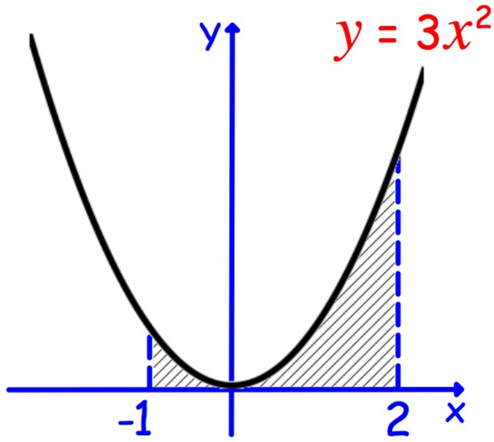
(1) $\int_{-1}^{2} 3x^2\,dx$

$= \left[x^3\right]_{-1}^{2}$

$= 2^3 - (-1)^3$

$= 8 - (-1)$

$= \underline{\underline{9}}$



Area 9 units^2

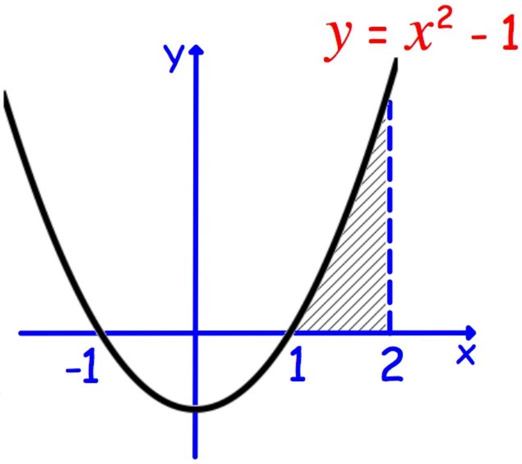
(2) $\int_{1}^{2} (x^2 - 1)\, dx$

$$= \left[\frac{x^3}{3} - x \right]_1^2$$

$$= \left(\frac{2^3}{3} - 2 \right) - \left(\frac{1^3}{3} - 1 \right)$$

$$= \frac{2}{3} - \left(-\frac{2}{3} \right)$$

$$= \frac{4}{3}$$



Area $\frac{4}{3}$ units^2

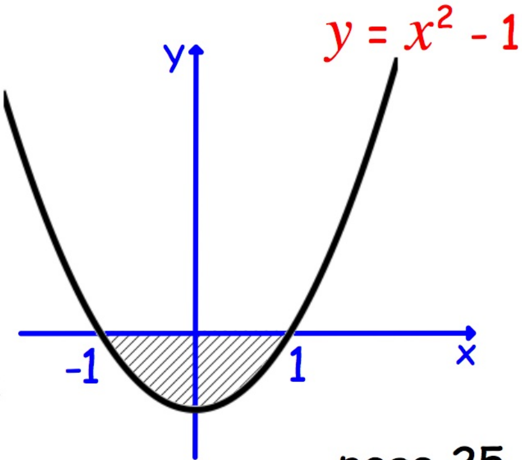
(3) $\int_{-1}^{1} (x^2 - 1)\, dx$

$$= \left[\frac{x^3}{3} - x \right]_{-1}^{1}$$

$$= \left(\frac{1^3}{3} - 1 \right) - \left(\frac{(-1)^3}{3} - (-1) \right)$$

$$= -\frac{2}{3} - \frac{2}{3}$$

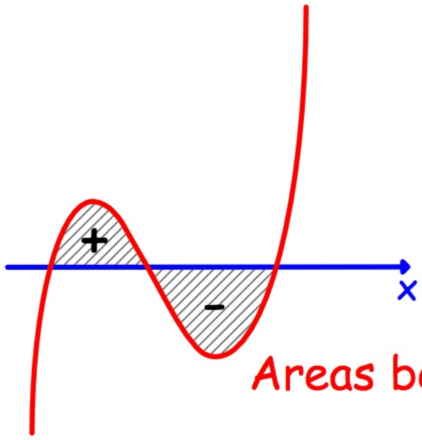
$$= -\frac{4}{3}$$



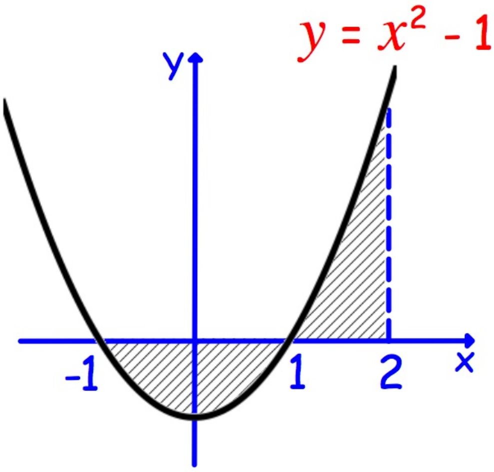
area cannot be negative,

Area $\frac{4}{3}$ units^2

Evaluate areas above and below the x-axis separately.



Areas below the x-axis appear to be negative.



$$\int_{-1}^{2} (x^2 - 1)\, dx = 0$$

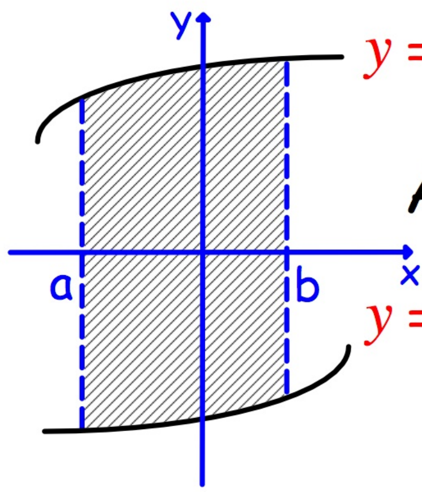
area below x-axis appears negative and cancels with area above x-axis

$$\int_{1}^{2} (x^2 - 1)\, dx = \frac{4}{3}$$

$$\int_{-1}^{1} (x^2 - 1)\, dx = -\frac{4}{3}$$

$$\text{Area} = \frac{4}{3} + \frac{4}{3} = \frac{8}{3} \text{ units}^2$$

AREA BETWEEN CURVES

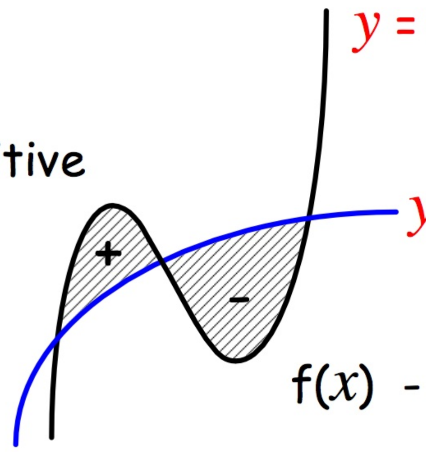


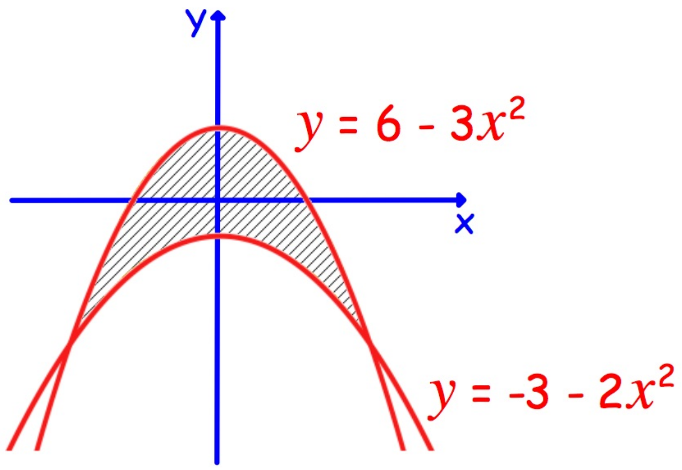
$$\text{AREA} = \int_a^b \{ f(x) - g(x) \} \, dx$$

NOTE: TOP - BOTTOM always gives a positive area

ie. area is NOT negative below the x-axis.

If the curves cross over evaluate the areas separately.





LIMITS: intersection

TOP CURVE = BOTTOM CURVE

$$6 - 3x^2 = -3 - 2x^2$$
$$9 - x^2 = 0$$
$$(3 + x)(3 - x) = 0$$
$$x = -3 \text{ or } 3$$

INTEGRAND

TOP - BOTTOM

$$6 - 3x^2 - (-3 - 2x^2)$$
$$= 9 - x^2$$

from the symmetry:

$$\int_{-3}^{3} (9 - x^2)\, dx = 2\int_{0}^{3} (9 - x^2)\, dx$$

easier to evaluate with limit 0

$$\int_{0}^{3} (9 - x^2)\, dx$$
$$= \left[9x - \frac{x^3}{3} \right]_0^3$$
$$= 9 \times 3 - \frac{3^3}{3} - 0$$
$$= 18$$

Area = 2 × 18 units^2 = 36 units^2

TRIGONOMETRY: COMPOUND ANGLES

EXPANSION FORMULAE

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

NOTICE: $\cos(A + B) \neq \cos A + \cos B$

$\cos A \cos B + \sin A \sin B = \cos(A - B)$

$$\begin{aligned}\cos 110^\circ \cos 20^\circ + \sin 110^\circ \sin 20^\circ &= \cos(110^\circ - 20^\circ)\\ &= \cos 90^\circ\\ &= 0\end{aligned}$$

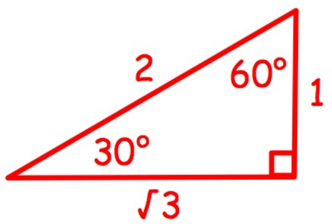
$\sin A \cos B + \cos A \sin B = \sin(A + B)$

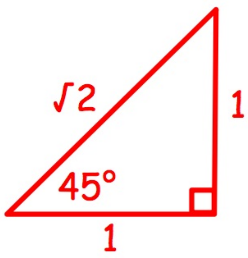
$$\begin{aligned}\sin 20^\circ \cos 10^\circ + \cos 20^\circ \sin 10^\circ &= \sin(20^\circ + 10^\circ)\\ &= \sin 30^\circ\\ &= {}^1/_2\end{aligned}$$

EXACT VALUES

No calculator allowed.

Use combinations of 30°, 45°, 60°





Find the EXACT value of cos 105°.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(60° + 45°) = \cos 60° \cos 45° - \sin 60° \sin 45°$$

$$= \tfrac{1}{2} \times \tfrac{1}{\sqrt{2}} - \tfrac{\sqrt{3}}{2} \times \tfrac{1}{\sqrt{2}}$$

$$= \tfrac{1}{2\sqrt{2}} - \tfrac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1 - \sqrt{3}}{2\sqrt{2}}$$

rationalising denominator:

$$\frac{1 - \sqrt{3}}{2\sqrt{2}} \quad \begin{matrix} \times \sqrt{2} \\ \times \sqrt{2} \end{matrix}$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

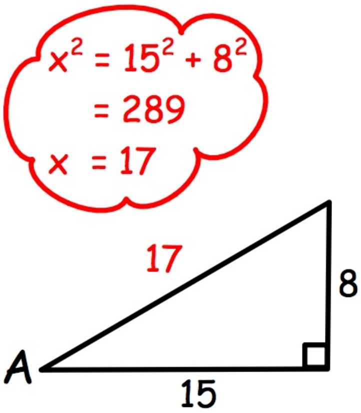
USING TRIG. RATIOS

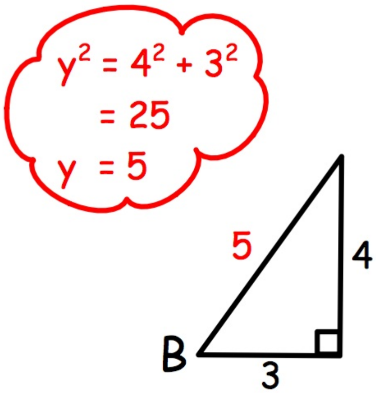
SOHACHTOA to sketch triangles

PYTH. THM. to find missing side

Find values of sin and cos of angle

If $\tan A = {}^{8}/_{15}$ and $\tan B = {}^{4}/_{3}$, find $\cos(A+B)$.





$\sin A = {}^{8}/_{17}$ $\quad\quad$ $\sin B = {}^{4}/_{5}$

$\cos A = {}^{15}/_{17}$ $\quad\quad$ $\cos B = {}^{3}/_{5}$

$$\begin{aligned}
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
&= {}^{15}/_{17} \times {}^{3}/_{5} - {}^{8}/_{17} \times {}^{4}/_{5} \\
&= {}^{45}/_{85} - {}^{32}/_{85} \\
&= {}^{13}/_{85}
\end{aligned}$$

DOUBLE ANGLE FORMULAE

$\sin 2A = 2\sin A \cos A$

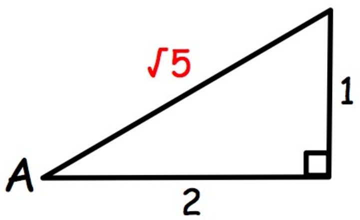
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

VARIATIONS:

$$\sin 2A = 2\sin A \cos A$$
$$\sin 4A = 2\sin 2A \cos 2A$$
$$\sin A = 2\sin {}^{A}/_{2} \cos {}^{A}/_{2}$$

(1) If $\tan A = {}^{1}/_{2}$, find $\sin 2A$.

$$x^2 = 2^2 + 1^2$$
$$= 5$$
$$x = \sqrt{5}$$



$\sin A = {}^{1}/_{\sqrt{5}}$

$\cos A = {}^{2}/_{\sqrt{5}}$

$$\sin 2A = 2\sin A \cos A$$
$$= {}^{2}/_{1} \times {}^{1}/_{\sqrt{5}} \times {}^{2}/_{\sqrt{5}}$$
$$= {}^{4}/_{5}$$

(2) If $\tan A = {}^{4}/_{3}$, find:

(i) $\sin 2A$ (ii) $\cos 2A$ (iii) $\sin 4A$ (iv) $\tan 2A$

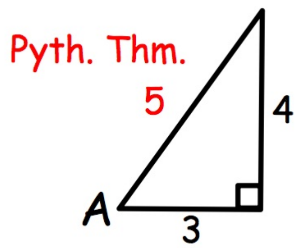
$\sin 2A = 2\sin A \cos A$

$= {}^{2}/_{1} \times {}^{4}/_{5} \times {}^{3}/_{5}$

$= {}^{24}/_{25}$

$\sin A = {}^{4}/_{5}$

$\cos A = {}^{3}/_{5}$



$\cos 2A = \cos^2 A - \sin^2 A$

$= ({}^{3}/_{5})^2 - ({}^{4}/_{5})^2$

$= {}^{9}/_{25} - {}^{16}/_{25}$

$= -{}^{7}/_{25}$

or can use:

$\cos 2A = 2\cos^2 A - 1$

$\cos 2A = 1 - 2\sin^2 A$

Using $\sin 2A = {}^{24}/_{25}$, $\cos 2A = -{}^{7}/_{25}$ and IDENTITIES:

$\sin 2A = 2\sin A \cos A$

$\tan A = \dfrac{\sin A}{\cos A}$

$\sin 4A = 2\sin 2A \cos 2A$

$= {}^{2}/_{1} \times {}^{24}/_{25} \times (-{}^{7}/_{25})$

$= -{}^{336}/_{625}$

$\tan 2A = \dfrac{\sin 2A}{\cos 2A}$

$= \dfrac{{}^{24}/_{25}}{-{}^{7}/_{25}}$

$= {}^{24}/_{-7}$

$= -{}^{24}/_{7}$

TRIG. EQUATIONS: DOUBLE ANGLES

Equation contains: $\sin 2x$ or $\cos 2x$

AND $\sin x$ or $\cos x$

REPLACEMENTS: $\sin 2x$ by $2\sin x \cos x$

if equation has a $\cos x$ — $\cos 2x$ by $2\cos^2 x - 1$

if equation has a $\sin x$ — $\cos 2x$ by $1 - 2\sin^2 x$

FACTORISE: form $ax^2 + bx + c = 0$

$(\quad)(\quad) = 0$

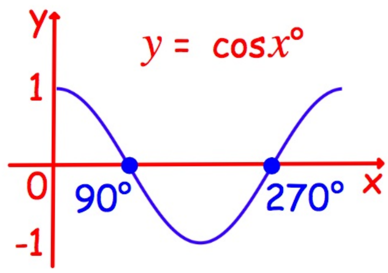
SOLVE: Trig. equations using CAST or graphs.

(1) Solve $\sin 2x° - 3\cos x° = 0$, $0 \leq x \leq 360$

$2\sin x° \cos x° - 3\cos x° = 0$

$2\sin x° \cos x° - 3\cos x° = 0$

$\cos x° (2\sin x° - 3) = 0$



$2\sin x° - 3 = 0$ or $\cos x° = 0$

$2\sin x° = 3$

$\sin x° = {}^3/_2$

no solution $\qquad x = 90$ or 270

$\underline{\underline{x = 90, 270}}$

(2) Solve $\cos 2x^\circ - 3\cos x^\circ + 2 = 0$, $0 \leq x \leq 360$

$$2\cos^2 x^\circ - 1 - 3\cos x^\circ + 2 = 0$$
$$2\cos^2 x^\circ - 3\cos x^\circ + 1 = 0$$
$$(2\cos x^\circ - 1)(\cos x^\circ - 1) = 0$$

$2\cos x^\circ - 1 = 0$ or $\cos x^\circ - 1 = 0$

$2\cos x^\circ = 1$

$\cos x^\circ = {}^1/_2$ or $\cos x^\circ = 1$

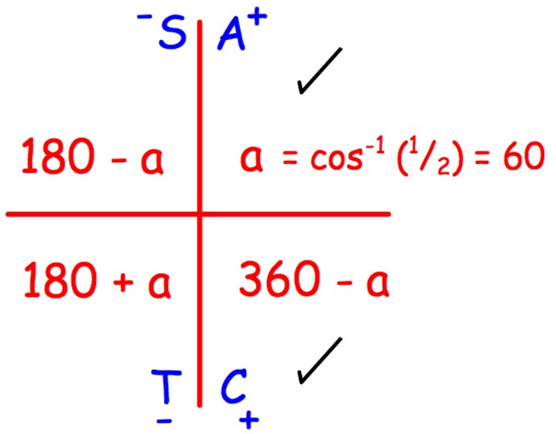
$x = 60$ or 300 $x = 0$ or 360

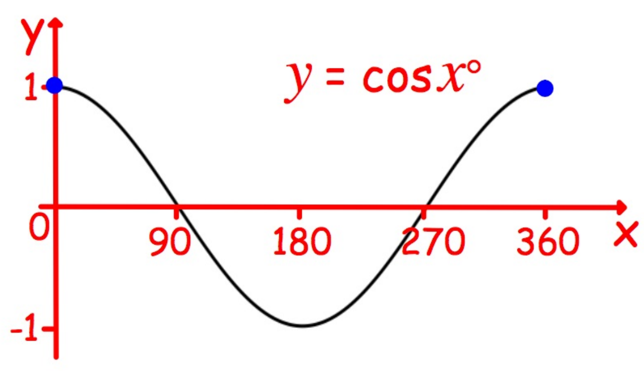
$$\underline{\underline{x = 0, 60, 300, 360}}$$

in radians: $\cos 2x - 3\cos x + 2 = 0$, $0 \leq x \leq 2\pi$

$$\underline{\underline{x = 0, {}^{\pi}/_3, {}^{5\pi}/_3, 2\pi}}$$

REMINDERS:





0° , 360°

$a = 60$

$360 - a = 300$

TRIG. FORMULAE: PROOFS

REPLACEMENTS may be made using the identities:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \sin^2 A - \cos^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

OTHER FORMULAE:

$$\sin^2 A + \cos^2 A = 1$$
$$\cos^2 A = 1 - \sin^2 A$$
$$\sin^2 A = 1 - \cos^2 A$$

$$\tan A = \frac{\sin A}{\cos A}$$

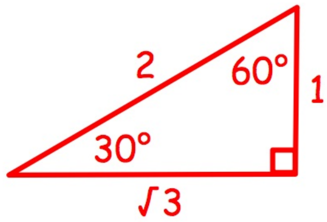
Sine Rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

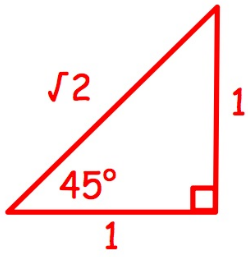
Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Area Formula: area $\Delta ABC = \frac{1}{2}ab\sin C$

EXACT VALUES:

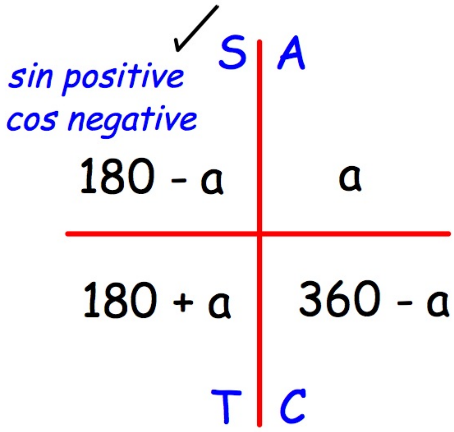




REMINDERS:

$\cos 120°$
$= \cos(180 - 60)°$
$= -\cos 60°$
$= -{}^{1}/_{2}$

$\sin 120°$
$= \sin(180 - 60)°$
$= +\sin 60°$
$= {}^{\sqrt{3}}/_{2}$



(1) Show that $\cos(x-30)° + \sin(x+120)° = \sqrt{3}\cos x°$

$\cos(x-30)° \quad + \quad \sin(x+120)°$

$= \cos x°\cos 30° + \sin x°\sin 30°$
$\quad + \sin x°\cos 120° + \cos x°\sin 120°$

$= \cos x° \times {}^{\sqrt{3}}/_{2} + \sin x° \times {}^{1}/_{2}$
$\quad + \sin x° \times (-{}^{1}/_{2}) + \cos x° \times {}^{\sqrt{3}}/_{2}$

$= {}^{\sqrt{3}}/_{2}\cos x° + {}^{1}/_{2}\sin x° - {}^{1}/_{2}\sin x° + {}^{\sqrt{3}}/_{2}\cos x°$

$= \underline{\underline{\sqrt{3}\cos x°}}$

(2) Show $(1 - \sin^2 A)(1 - \tan^2 A) = \cos 2A$

$$(1 - \sin^2 A)(1 - \tan^2 A) = \cos^2 A\left(1 - \frac{\sin^2 A}{\cos^2 A}\right)$$

$$= \cos^2 A - \sin^2 A$$

$$= \underline{\underline{\cos 2A}}$$

(3) Use $\cos 2A$ identities to show that

$\sin^2 A = {}^1/_2(1 - \cos 2A)$ and $\cos^2 A = {}^1/_2(1 + \cos 2A)$

and hence show $\sin^2 A + 5\cos^2 A = 3 + 2\cos 2A$

$1 - 2\sin^2 A = \cos 2A$

$2\sin^2 A = 1 - \cos 2A$

$\sin^2 A = {}^1/_2(1 - \cos 2A)$

$2\cos^2 A - 1 = \cos 2A$

$2\cos^2 A = 1 + \cos 2A$

$\cos^2 A = {}^1/_2(1 + \cos 2A)$

$$\sin^2 A + 5\cos^2 A$$

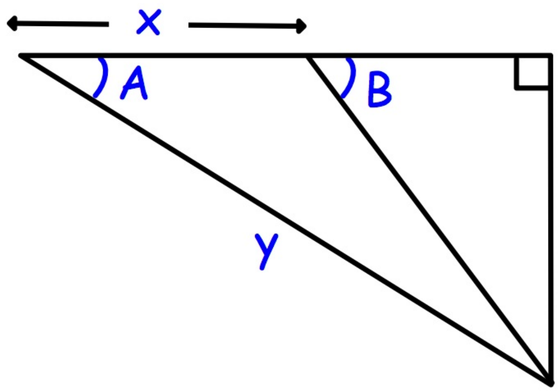
$$= {}^1/_2(1 - \cos 2A) + 5 \times {}^1/_2(1 + \cos 2A)$$

$$= {}^1/_2 - {}^1/_2\cos 2A + {}^5/_2 + {}^5/_2\cos 2A$$

$$= \underline{\underline{3 + 2\cos 2A}}$$

(4) Show that

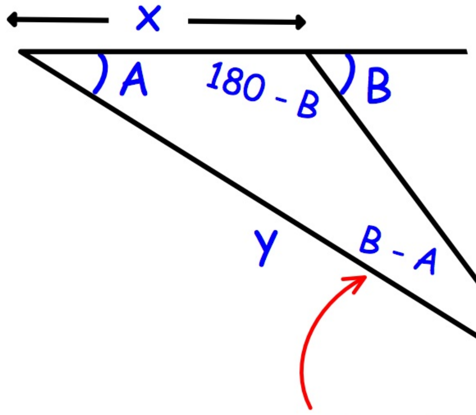
$$x = \frac{y\sin(B-A)}{\sin B}$$



SINE RULE:

$$\frac{x}{\sin(B-A)} = \frac{y}{\sin(180-B)}$$

$$x = \frac{y\sin(B-A)}{\sin B}$$



$180 - (A + 180 - B)$
$= 180 - A - 180 + B$
$= B - A$

$\sin(180 - B) = +\sin B$