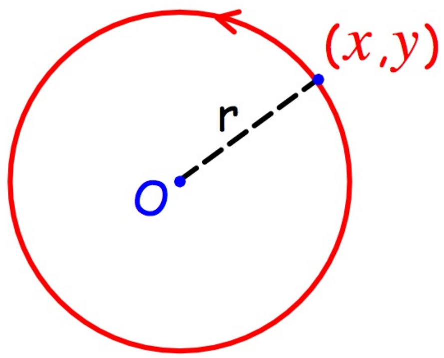
CIRCLES

LOCUS: all points r units from O.

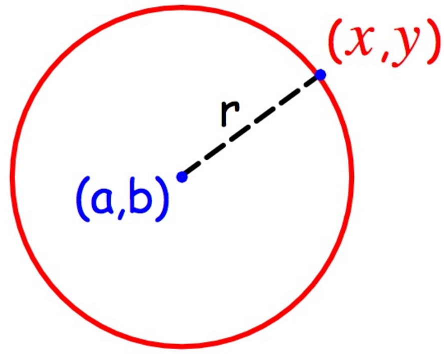
CIRCLE centre (0,0), radius r.

$$x^2 + y^2 = r^2$$



CIRCLE centre (a,b), radius r.

$$(x - a)^2 + (y - b)^2 = r^2$$



(1) Find the equation of the circle:

(a) centre (0,0) and passing through point (-2,1)

$$x^2 + y^2 = r^2$$

$$(-2)^2 + 1^2 = r^2$$

$$r^2 = 5$$

$$\underline{\underline{x^2 + y^2 = 5}}$$

(b) twice the radius and centre (2,-5)

$$r = \sqrt{5}$$

$$2r = 2\sqrt{5} = \sqrt{4} \times \sqrt{5} = \sqrt{20}$$

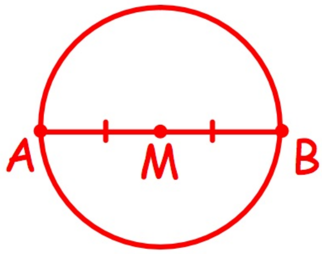
$$(x - a)^2 + (y - b)^2 = r^2$$

$(\overset{a}{2}, \overset{b}{-5})$

$$(x - 2)^2 + (y + 5)^2 = (\sqrt{20})^2$$

$$\underline{\underline{(x - 2)^2 + (y + 5)^2 = 20}}$$

(2) Find the equation of the circle with diametrically opposite points A(-4,1) and B(2,3).



M_{AB} (-1,2)

$$(x - a)^2 + (y - b)^2 = r^2$$

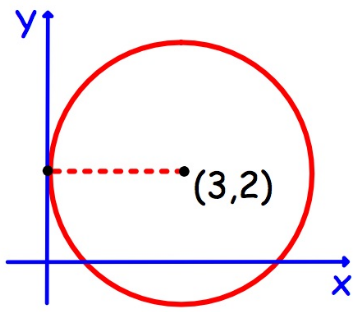
$(\overset{a}{-1}, \overset{b}{2})$ $\quad (x + 1)^2 + (y - 2)^2 = r^2$

$B(\overset{x}{2}, \overset{y}{3})$ $\quad (2 + 1)^2 + (3 - 2)^2 = r^2$

or use A(-4,1) $\quad r^2 = 10$

$$\underline{\underline{(x + 1)^2 + (y - 2)^2 = 10}}$$

(3) Is the point (5,-1) inside, outside or on this circle?



$(\overset{a}{3}, \overset{b}{2})$ $\quad (x - a)^2 + (y - b)^2 = r^2$

$r = 3$ $\quad (x - 3)^2 + (y - 2)^2 = 9$

$(\overset{x}{5}, \overset{y}{-1})$

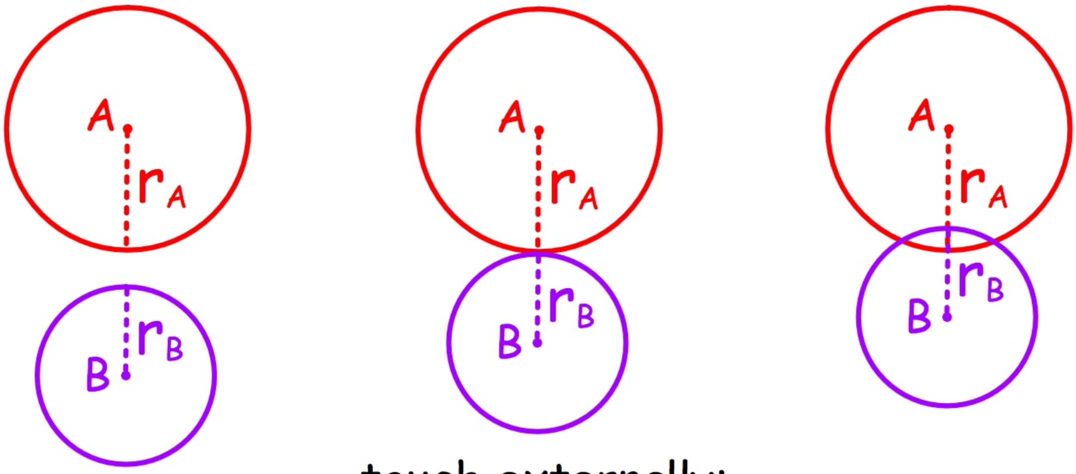
$$\begin{aligned} &(x - 3)^2 + (y - 2)^2 \\ &= (5 - 3)^2 + (-1 - 2)^2 \\ &= 13 \end{aligned}$$

$$(x - 3)^2 + (y - 2)^2 > 9$$

$\Rightarrow$ <u>point outside circle</u>

INTERSECTING CIRCLES

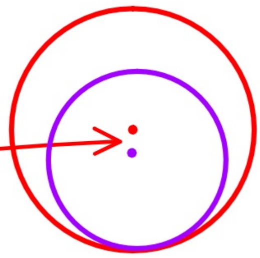
Difference between ($r_A + r_B$) and AB is the gap or overlap.



touch externally:

$r_A + r_B = AB$

touch internally: $r_A - r_B = AB$



Show that circles $x^2 + y^2 = 4$ and $(x - 3)^2 + (y - 4)^2 = 9$ touch externally.

$r_A = 2$	$r_B = 3$
A (0,0)	B (3,4)

distance between centres:

$$AB^2 = (3 - 0)^2 + (4 - 0)^2$$
$$= 25$$
$$AB = 5$$

sum of radii:

$$r_A + r_B$$
$$= 2 + 3$$
$$= 5$$

$AB = r_A + r_B$

$\Rightarrow$ circles touch externally

GENERAL EQUATION OF A CIRCLE

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

centre $(-g, -f)$ radius $\sqrt{g^2 + f^2 - c}$

so requires $g^2 + f^2 - c > 0$ ie. positive

(1) If $x^2 + y^2 - 2x + 8y + k = 0$ is a circle, find the possible values of k.

$$x^2 + y^2 - 2x + 8y + k = 0$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$g = -1$ $f = 4$ $c = k$

$g^2 + f^2 - c$

$= (-1)^2 + 4^2 - k$

$= 17 - k$

for a circle $g^2 + f^2 - c > 0$

$17 - k > 0$

$-k > -17$

$\underline{\underline{k < 17}}$

(2) Find the centre and radius of circle
$x^2 + y^2 - 2x + 6y - 15 = 0$

$x^2 + y^2 - 2x + 6y - 15 = 0$
$x^2 + y^2 + 2gx + 2fy + c = 0$

$g = -1 \quad f = 3 \quad c = -15$

$g^2 + f^2 - c$
$= (-1)^2 + 3^2 - (-15)$
$= 25$

$r = \sqrt{g^2 + f^2 - c} = \sqrt{25} = 5$

centre $(\overset{-g}{1}, \overset{-f}{-3})$, radius 5 units

(3) Find the centre and radius of circle
$x^2 + y^2 - 10y + 7 = 0$

$x^2 + y^2 + 0x - 10y + 7 = 0$
$x^2 + y^2 + 2gx + 2fy + c = 0$

$g = 0 \quad f = -5 \quad c = 7$

$g^2 + f^2 - c$
$= 0^2 + (-5)^2 - 7$
$= 18$

$r = \sqrt{g^2 + f^2 - c} = \sqrt{18} = 3\sqrt{2}$

centre $(\overset{-g}{0}, \overset{-f}{5})$, radius $3\sqrt{2}$ units

NOTE:

GIVEN equation $x^2 + y^2 + 2gx + 2fy + c = 0$
FIND centre and radius

GIVEN centre and radius
FIND equation using $(x - a)^2 + (y - b)^2 = r^2$

Find the equation of the circle CONCENTRIC with circle $x^2 + y^2 + 6x - 4y - 3 = 0$ but with twice the radius.

$x^2 + y^2 + 6x - 4y - 3 = 0$

$x^2 + y^2 + 2gx + 2fy + c = 0$

$g = 3 \quad f = -2 \quad c = -3$

$g^2 + f^2 - c$
$= 3^2 + (-2)^2 - (-3)$
$= 16$

$r = \sqrt{g^2 + f^2 - c} = \sqrt{16} = 4$

centre $(\overset{-g}{-3}, \overset{-f}{2})$, radius 4 units

centre $(\overset{a}{-3}, \overset{b}{2})$, radius 8 units

$(x - a)^2 + (y - b)^2 = r^2$

$\underline{\underline{(x + 3)^2 + (y - 2)^2 = 64}}$

LENGTH OF A TANGENT

(i) radius and centre from equation of circle

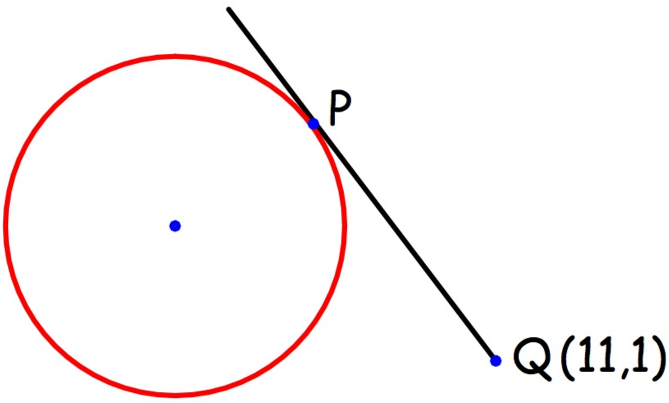
(ii) distance from centre by distance formula

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

(iii) length of tangent by Pyth. Thm.

Find the distance PQ.

$$x^2 + (y - 3)^2 = 25$$

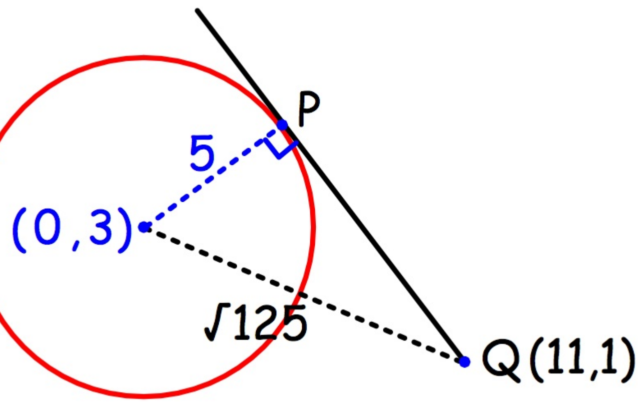


$$x^2 + (y - 3)^2 = 25$$

$$(x - a)^2 + (y - b)^2 = r^2$$

centre $(\overset{a}{0},\overset{b}{3})$

radius 5



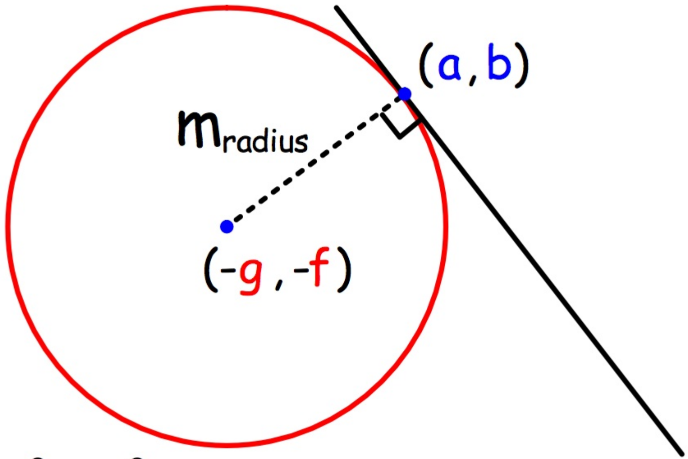
Distance formula

$$\begin{aligned} d^2 &= (11 - 0)^2 + (1 - 3)^2 \\ &= 11^2 + (-2)^2 \\ &= 125 \end{aligned}$$

Pyth. Thm.

$$\begin{aligned} PQ^2 &= d^2 - r^2 \\ &= 125 - 5^2 \\ &= 100 \\ PQ &= 10 \text{ units} \end{aligned}$$

EQUATION OF A TANGENT



$$m_{tangent} \times m_{radius} = -1$$

$m_{tangent}$

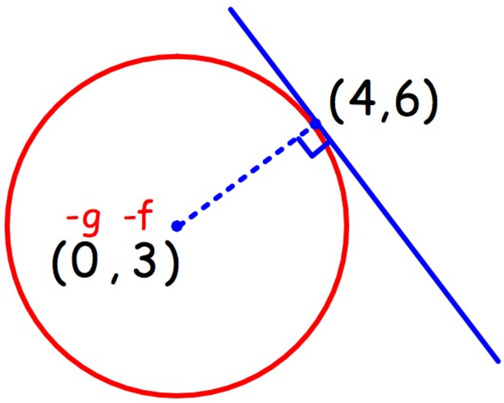
$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad y - b = m(x - a)$$

Find the equation of the tangent which meets the circle $x^2 + y^2 - 6y - 16 = 0$ at the point (4,6).

$$x^2 + y^2 + 0x - 6y - 16 = 0$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$g = 0 \qquad f = -3$$



$$m_{radius} = \frac{6-3}{4-0} = \frac{3}{4}$$

ppn gradient: $m_1 \times m_2 = -1$

$$m_{tangent} = -\frac{4}{3}$$

$$y - b = m(x - a)$$

P(4,6) $\qquad y - 6 = \frac{-4}{3}(x - 4)$

$$3y - 18 = -4x + 16$$

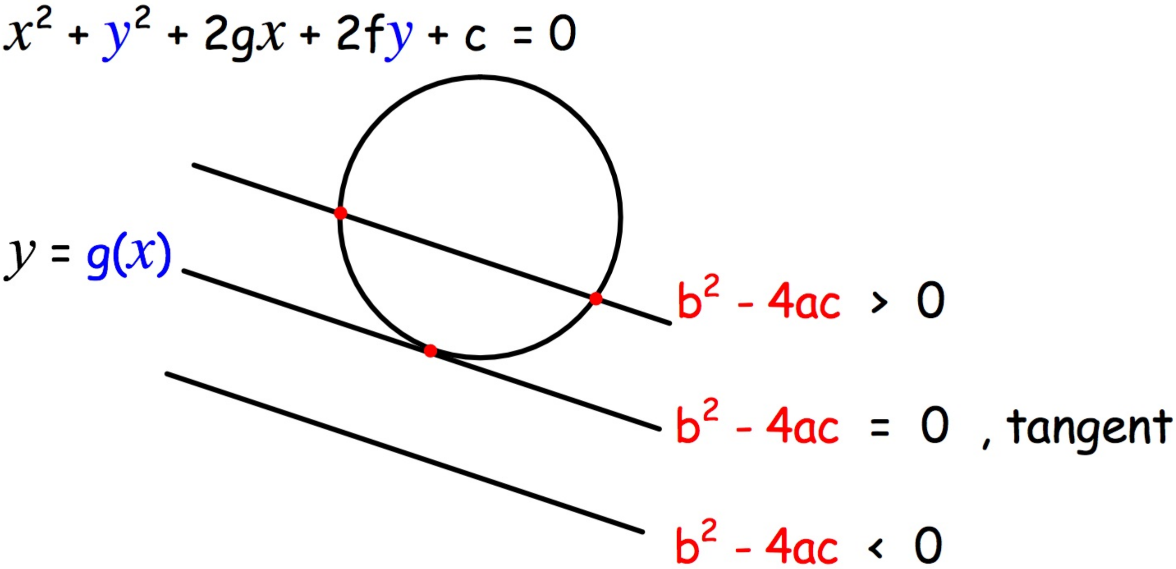
$$\underline{\underline{4x + 3y = 34}}$$

INTERSECTION OF A LINE AND CIRCLE

Substitute $y = g(x)$ into $x^2 + y^2 + 2gx + 2fy + c = 0$

results in $ax^2 + bx + c = 0$

Discriminant $b^2 - 4ac$ distinguishes between:



TANGENCY

A tangent is a line that touches the circle at one point.

The substitution results in a quadratic equation
with one solution:

$b^2 - 4ac = 0 \quad \Rightarrow$ EQUAL ROOTS
so line is a tangent

Show that the line $x + y = 4$ is a tangent to the circle $x^2 + y^2 + 6x + 2y - 22 = 0$ and find the point of contact.

$$x + y = 4$$

substitution: $y = 4 - x$

$$x^2 + y^2 + 6x + 2y - 22 = 0$$

$$x^2 + (4 - x)^2 + 6x + 2(4 - x) - 22 = 0$$

$$x^2 + 16 - 8x + x^2 + 6x + 8 - 2x - 22 = 0$$

$$2x^2 - 4x + 2 = 0$$

simplify quadratic equation: $x^2 - 2x + 1 = 0$

discriminant: $1x^2 - 2x + 1 = 0$

$a = 1$, $b = -2$, $c = 1$

$$b^2 - 4ac = (-2)^2 - 4 \times 1 \times 1 = 0$$

* $b^2 - 4ac = 0 \Rightarrow$ line is a tangent

$x^2 - 2x + 1 = 0$

$(x - 1)^2 = 0$

$x = 1$

$y = 4 - x$

$= 4 - 1$

$= 3$

point of contact (1,3)

* OR

one point of contact $\Rightarrow$ line is a tangent