HIGHER MATHEMATICS
COURSE NOTES

UNIT 1

FORMULAE LIST

Circle:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

The equation $(x - a)^2 + (y - b)^2 = r^2$ represents a circle centre (a, b) and radius r.

Scalar Product: $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$

or $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Trigonometric formulae:

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \sin 2A &= 2\sin A \cos A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}$$

Table of standard derivatives:

$f(x)$	$f'(x)$
$\sin ax$	$a \cos ax$
$\cos ax$	$-a \sin ax$

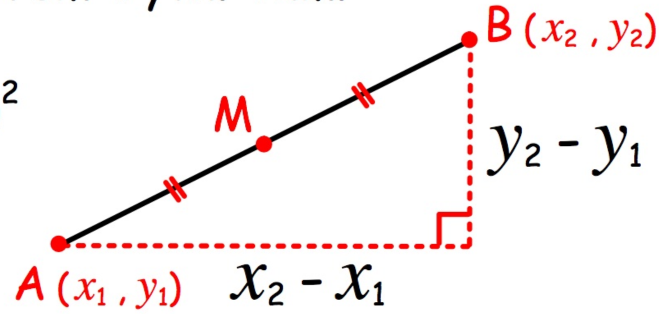
Table of standard integrals:

$f(x)$	$\int f(x)\,dx$
$\sin ax$	$-\frac{1}{a}\cos ax + C$
$\cos ax$	$\frac{1}{a}\sin ax + C$

STRAIGHT LINE

DISTANCE FORMULA from Pyth. Thm.

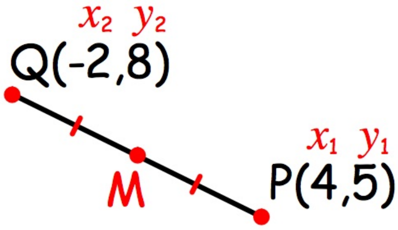
$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$



MID-POINT

$$M_{AB}\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

Points P(4,5) and Q(-2,8).



$$PQ^2 = (\overset{x_2}{-2} - \overset{x_1}{4})^2 + (\overset{y_2}{8} - \overset{y_1}{5})^2$$

$$= (-6)^2 + 3^2$$

$$= 36 + 9$$

$$= 45$$

$$PQ = \sqrt{45}$$

$$= \sqrt{9} \times \sqrt{5}$$

$$\underline{\underline{PQ = 3\sqrt{5} \text{ units}}}$$

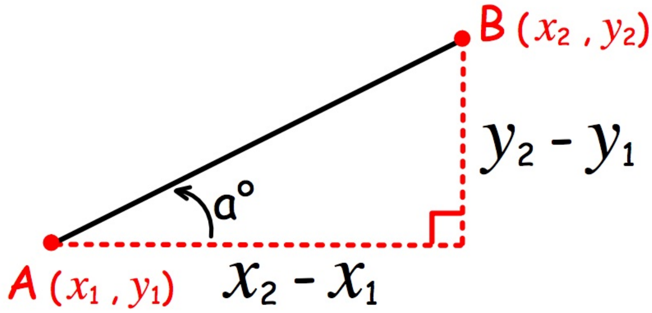
$$\frac{\overset{x_1}{4} + \overset{x_2}{(-2)}}{2} \qquad \frac{\overset{y_1}{5} + \overset{y_2}{8}}{2}$$

$$\underline{\underline{M_{PQ}\left(1, \frac{13}{2}\right)}}$$

Note: same result whichever is the first point.

GRADIENT FORMULA

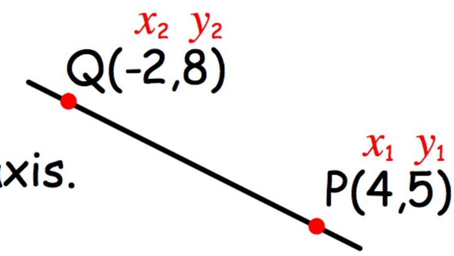
$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} \quad , \; x_2 \neq x_1$$



$m_{AB} =$ tan a° , $0 < a < 180$

a° is the anti-clockwise angle with the positive OX direction

Find the angle the line through P(4,5) and Q(-2,8) makes with the x-axis.



$$m_{PQ} = \frac{\overset{y_2}{8} - \overset{y_1}{5}}{\underset{x_2}{-2} - \underset{x_1}{4}}$$

$$= \frac{3}{-6}$$

$$= -\frac{1}{2}$$

$$\frac{5 - 8}{4 - (-2)}$$

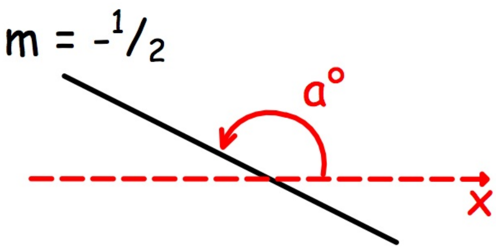
NOTE: same result whichever is the first point.

$$\text{tan a}° = -\frac{1}{2}$$

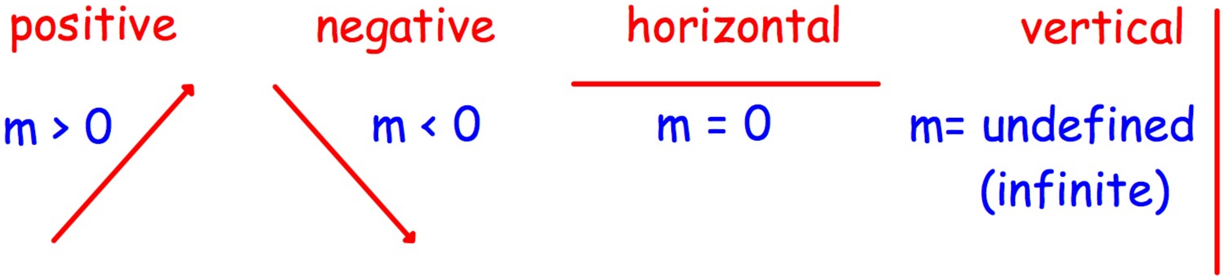
$\tan^{-1}(^1/_2)$

a = 180 - 26.565....

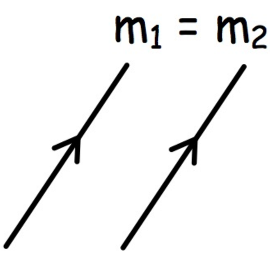
<u>angle 153.4°</u>

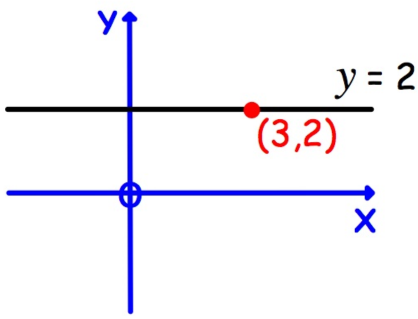


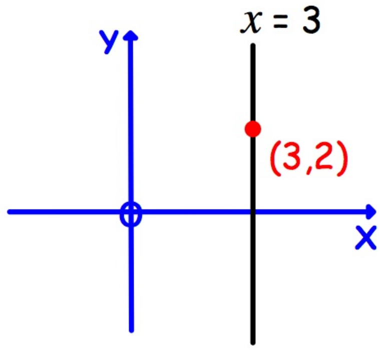
NOTE:



parallel lines: equations:



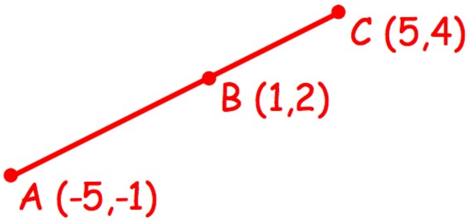




COLLINEARITY

Collinear points lie on the same straight line.

Show that the points A, B and C are collinear.



$$m_{AB} = \frac{2-(-1)}{1-(-5)} = \frac{3}{6} = \frac{1}{2}$$

$$m_{BC} = \frac{4-2}{5-1} = \frac{2}{4} = \frac{1}{2}$$

$m_{AB} = m_{BC}$

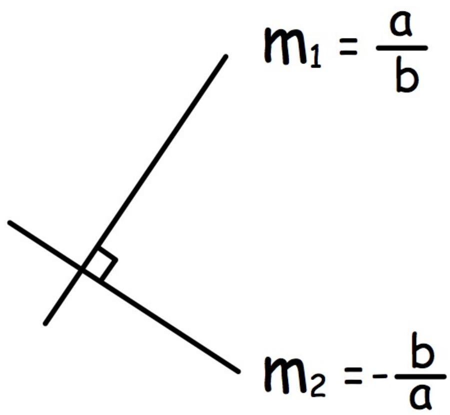
$\Rightarrow$ AB is parallel to BC

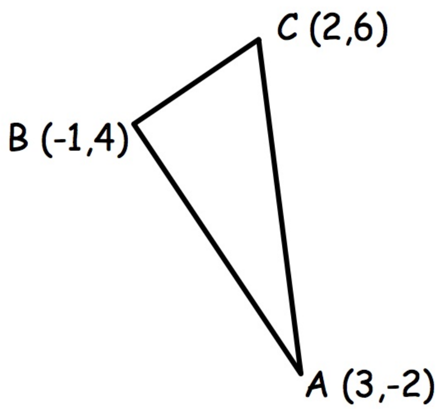
and since lines share point B

$\Rightarrow$ points A, B and C are collinear

PERPENDICULAR LINES

$m_1 \times m_2 = -1$





Show that ΔABC is right-angled at B.

$$m_{AB} = \frac{4-(-2)}{-1-3} = \frac{6}{-4} = -\frac{3}{2}$$

$$m_{BC} = \frac{6-4}{2-(-1)} = \frac{2}{3}$$

$$m_{AB} \times m_{BC} = -\frac{3}{2} \times \frac{2}{3} = -1$$

$m_{AB} \times m_{BC} = -1$

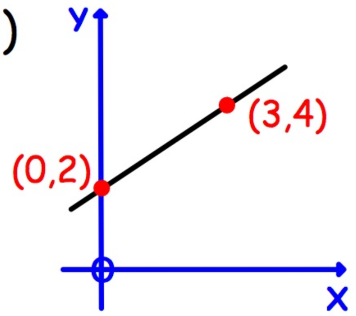
$\Rightarrow$ AB is perpendicular BC

$\Rightarrow \angle ABC = 90°$

EQUATION OF A LINE

Gradient m and Y-intercept C : $y = mx + C$

(1)



$$m = \frac{4-2}{3-0} = \frac{2}{3}$$

$C = 2$

$y = mx + C$

$$y = \frac{2}{3}x + 2$$

x3 to remove fraction

$3y = 2x + 6$

$0 = 2x - 3y + 6$

Can also be written in the form $Ax + By + C = 0$

$$2x - 3y + 6 = 0$$

(2) Find the gradient of the line $2x - 3y - 9 = 0$.

rearrange to $y = mx + C$

$2x - 9 = 3y$

$\frac{2}{3}x - 3 = y$

$y = \frac{2}{3}x - 3$

$$m = \frac{2}{3}$$

(3) If point (k,4) lies on line $2x + 3y = 6$, find k.

$x = k$, $y = 4$

$2 \times k + 3 \times 4 = 6$

$2k = -6$

$$k = -3$$

EQUATION OF A LINE

Gradient m and through point (a,b) : $y - b = m(x - a)$

(1) Equation of the line gradient $^{-2}/_3$, through P(-1,2).

$$y - b = m(x - a)$$

$P(\overset{a}{-1}, \overset{b}{2})$

$m = {}^{-2}/_3$

$$y - 2 = \frac{-2}{3}(x - (-1))$$

remove fraction before breaking brackets

$$3(y - 2) = -2(x + 1)$$

multiplied both sides by 3

$$3y - 6 = -2x - 2$$

$$\underline{\underline{3y = -2x + 4}}$$

(2) Equation of the line through P(2,1) and Q(3,-1).

$P(\overset{x_2}{2}, \overset{y_2}{1})$

$Q(\overset{x_1}{3}, \overset{y_1}{-1})$

$$m_{PQ} = \frac{\overset{y_2}{1} - \overset{y_1}{(-1)}}{\underset{x_2}{2} - \underset{x_1}{3}} = \frac{2}{-1} = -2$$

$$y - b = m(x - a)$$

$P(\overset{a}{2}, \overset{b}{1})$

$m = -2$

$$y - 1 = -2(x - 2)$$

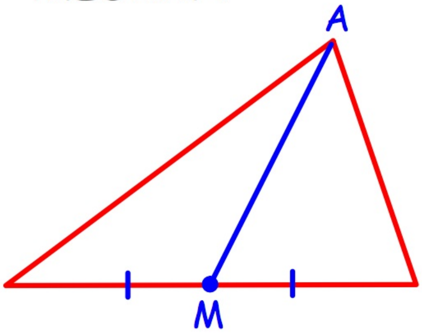
or can use Q(3,-1)

$$y - 1 = -2x + 4$$

$$\underline{\underline{y = -2x + 5}}$$

LINES ASSOCIATED WITH TRIANGLES

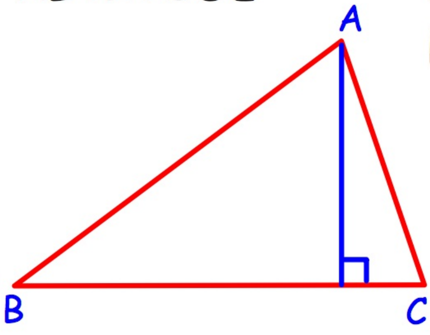
MEDIAN



FIND:
mid-point M

EQUATION:
use points A and M

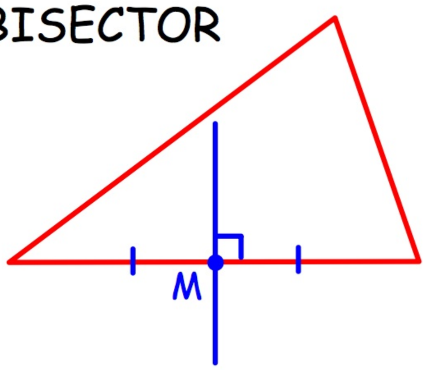
ALTITUDE



gradient BC
ppn. gradient
(by $m_1 \times m_2 = -1$)

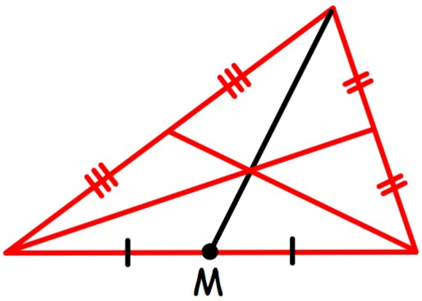
use point A
and ppn. gradient

PERPENDICULAR BISECTOR

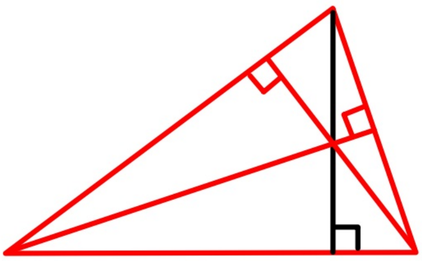


gradient BC
ppn. gradient
mid-point M

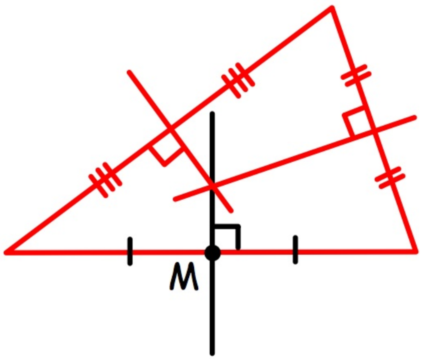
use point M
and ppn. gradient



the 3 medians are concurrent



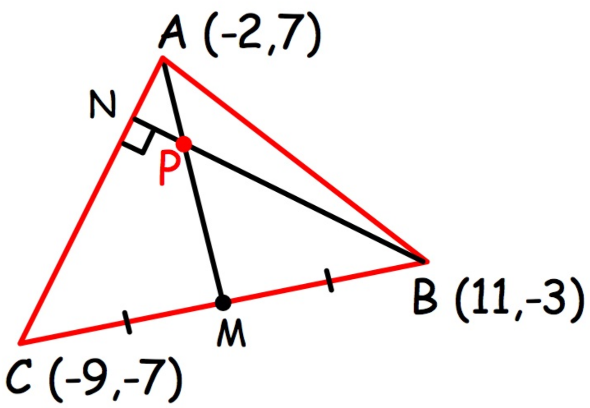
the 3 altitudes are concurrent



the 3 ppn bis are concurrent

Lines are concurrent if they pass through a common point.

Find P, the point of intersection of the median AM and altitude BN.



MEDIAN AM

midpoint: $M(\overset{x_1}{1}, \overset{y_1}{-5})$ $\quad \left(\frac{-9+11}{2}, \frac{-7+(-3)}{2}\right)$

gradient: $A(\overset{x_2}{-2}, \overset{y_2}{7})$

$$m_{AM} = \frac{\overset{y_2}{7} - \overset{y_1}{(-5)}}{\underset{x_2}{-2} - \underset{x_1}{1}} = \frac{12}{-3} = -4$$

line:

$A(\overset{a}{-2}, \overset{b}{7})$ $\quad m = -4$

$$y - b = m(x - a)$$

$$y - 7 = -4\,(x - (-2))$$

$$y - 7 = -4x - 8$$

$$\underline{\underline{4x + y = -1}}$$

ALTITUDE BN

gradient:

$$A(\overset{x_2}{-2}, \overset{y_2}{7}) \qquad C(\overset{x_1}{-9}, \overset{y_1}{-7})$$

$$m_{AC} = \frac{\overset{y_2}{7} - (\overset{y_1}{-7})}{\underset{x_2}{-2} - (\underset{x_1}{-9})} = \frac{14}{7} = 2$$

ppn. gradient: by $m_1 \times m_2 = -1$ $\qquad m_{BN} = -{}^1/_2$

line:

$A(\overset{a}{11}, \overset{b}{-3})$ $\quad m = -{}^1/_2$

$$y - b = m(x - a)$$

$$y - (-3) = \frac{-1}{2}(x - 11)$$

$$2y + 6 = -x + 11$$

$$\underline{\underline{x + 2y = 5}}$$

INTERSECTION

solve the system of equations

AM	$4x + y = -1$	$\times 2$
BN	$x + 2y = 5$	$\times (-1)$

$$8x + 2y = -2$$
$$-x - 2y = -5$$

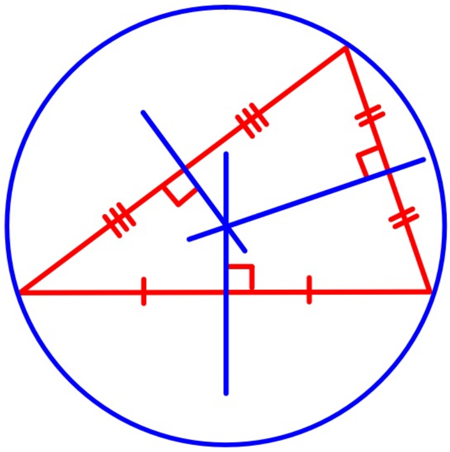
add

$$7x = -7$$
$$x = -1$$

$$4x + y = -1$$
$$-4 + y = -1$$
$$y = 3$$

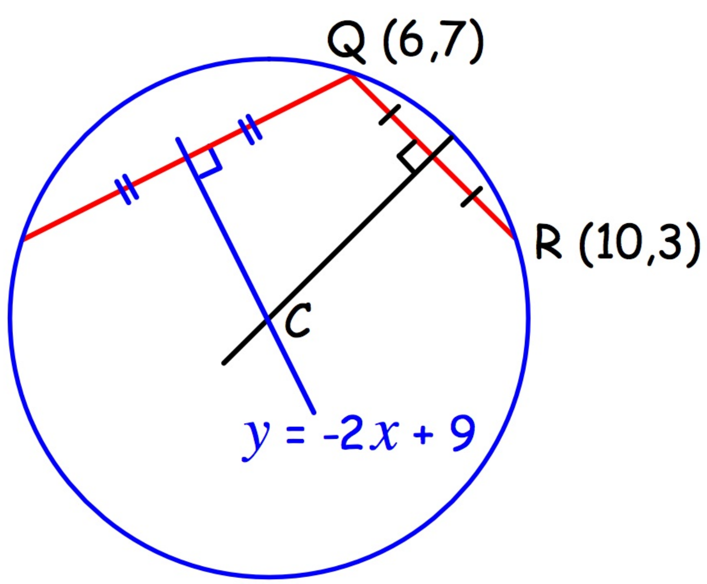
point of intersection $\underline{\underline{P(-1,3)}}$

CIRCLE



The 3 perpendicular bisectors are concurrent at the centre of the CIRCUMCIRCLE.

Find C, the centre of the circle.



PERPENDICULAR BISECTOR CM

gradient: $Q(\overset{x_2}{6}, \overset{y_2}{7})$ $R(\overset{x_1}{10}, \overset{y_1}{3})$

$$m_{QR} = \frac{\overset{y_2}{7} - \overset{y_1}{3}}{\underset{x_2}{6} - \underset{x_1}{10}} = \frac{4}{-4} = -1$$

ppn. gradient: by $m_1 \times m_2 = -1$ $m_{CM} = 1$

midpoint: $M_{QR}(8, 5)$

line:

$M(\overset{a}{8}, \overset{b}{5})$ $m = 1$

$$y - b = m(x - a)$$

$$y - 5 = 1(x - 8)$$

$$\underline{\underline{y = x - 3}}$$

INTERSECTION solve the system of equations

$$y = -2x + 9$$
$$y = x - 3$$

$$-2x + 9 = x - 3$$
$$-3x = -12$$
$$x = 4$$

$$y = x - 3$$
$$= 4 - 3$$
$$y = 1$$

centre $\underline{\underline{C\ (4,1)}}$

CONCURRENCY

Lines are concurrent if they pass through a common point.

FIND the point of intersection
SHOW this point lies on the other line

solve any pair of equations and show the point 'fits' the equation

Show that the lines with equations
$x + 3y = 10$, $2x - y = -1$ and $3x + y = 6$
are concurrent.

$x + 3y = 10$ $\times 1$
$2x - y = -1$ $\times 3$

$x + 3y = 10$
$6x - 3y = -3$

add $7x = 7$
$x = 1$

$x + 3y = 10$
$1 + 3y = 10$
$3y = 9$
$y = 3$

point of intersection (1,3)

third line $3x + y = 6$

(1,3) , $x = 1$ $3 \times 1 + y = 6$
$y = 3$

so (1,3) lies on the third line

All 3 lines pass through (1,3)

So lines concurrent at (1,3)

FUNCTIONS

Pairs members of one set of numbers with another.
Each DOMAIN element has a unique IMAGE.

This may require RESTRICTIONS on the domain.
If no domain specified - assume largest possible domain.

(1) $f(x) = \dfrac{1}{x+2}$ cannot $\div$ 0

not defined for $x = -2$

$\{ x : x \in \mathrm{R},\ x \neq -2 \}$

cannot $\sqrt{\text{negative}}$

(2) $f(x) = \sqrt{x-3}$

assume $+\sqrt{\ }$

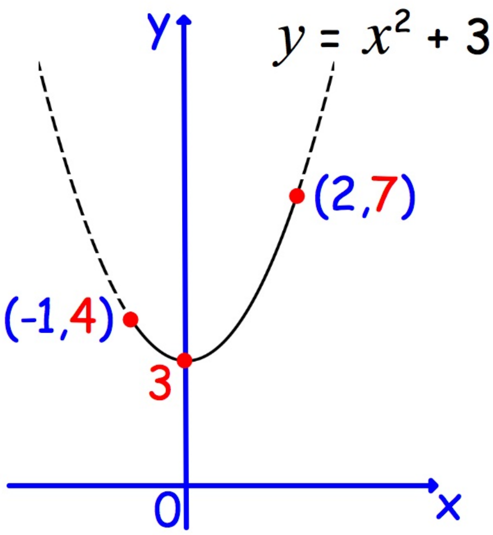
not defined for $x < 3$

$\{ x : x \in \mathrm{R},\ x \geq 3 \}$

The RANGE is the set of images.

Examine the graph and consider the domain specified.

Find the range of $f(x) = x^2 + 3$, $\{ x : x \in \mathrm{R},\ -1 \leq x \leq 2 \}$



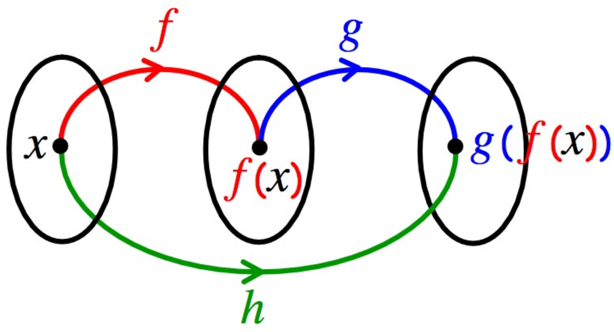
max. value 7

min. value 3

$3 \leq f(x) \leq 7$

Range $\underline{\underline{\{ y : y \in \mathrm{R},\ 3 \leq y \leq 7 \}}}$

COMPOSITE FUNCTIONS



The COMPOSITE function, $h(x) = g(f(x))$

acts second　　acts first

The order matters: generally $g(f(x)) \neq f(g(x))$

If $f(x) = x^2 - 2x$ and $g(x) = 1 - 2x$

(i) $g(f(x))$

$= g(x^2 - 2x)$

$= 1 - 2(x^2 - 2x)$

$= 1 - 2x^2 + 4x$

$= 1 + 4x - 2x^2$

(ii) $f(g(x))$

$= f(1 - 2x)$

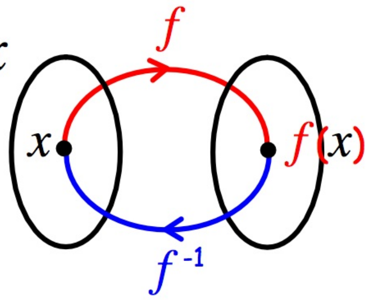
$= (1 - 2x)^2 - 2(1 - 2x)$

$= 1 - 4x + 4x^2 - 2 + 4x$

$= 4x^2 - 1$

INVERSE FUNCTIONS

$f^{-1}(f(x)) = x$



Requires a unique pairing:
one-to-one correspondence

$f(x) = x^2 + 2 \; , x \geq 0$ and $g(x) = \sqrt{x-2} \; , x \geq 2$

Find $f(g(x))$ and comment on the result.

$$
\begin{aligned}
& f(g(x)) \\
&= f(\sqrt{x-2}\,) \\
&= (\sqrt{x-2}\,)^2 + 2 \\
&= x - 2 + 2 \\
&= x
\end{aligned}
$$

$f(g(x)) = x$

$\Rightarrow$ f and g are inverse functions

Find the inverse function of $f(x) = 5x^3 + 3$

rearrange for x

$$y = 5x^3 + 3$$

$$5x^3 = y - 3$$

$$x^3 = \frac{y-3}{5}$$

$f^{-1}(y)$

$$x = \sqrt[3]{\frac{y-3}{5}}$$

change y to x

$$f^{-1}(x) = \sqrt[3]{\frac{x-3}{5}}$$

FUNCTIONS and GRAPHS

COMPLETING THE SQUARE: form $a(x \pm b)^2 + c$

Squaring brackets $(x \pm b)^2 = x^2 \pm 2bx + b^2$

For example, $(x + 3)^2 = x^2 + 6x + 9$

÷ 2 ÷ 2 and square

rearranging $x^2 \pm 2bx = (x \pm b)^2 - b^2$

$x^2 + 6x = (x + 3)^2 - 9$

÷ 2 square

÷ 2 and square

(1) $x^2 - 6x + 10$

$= (x - 3)^2 - 9 + 10$

$= (x - 3)^2 + 1$

÷ 2 and square

(2) $x^2 + 3x - 1$

$= (x + {}^3/_2)^2 - {}^9/_4 - {}^4/_4$

$= (x + {}^3/_2)^2 - {}^{13}/_4$

(3) $2x^2 - 12x + 10$

× 2

$= 2(x^2 - 6x) + 10$

$= 2(\,(x - 3)^2 - 9\,) + 10$

$= 2(x - 3)^2 - 18 + 10$

$= 2(x - 3)^2 - 8$

(4) $3 - 6x - x^2$

× -1

$= -1(x^2 + 6x) + 3$

$= -1(\,(x + 3)^2 - 9\,) + 3$

$= -1(x + 3)^2 + 9 + 3$

$= -1(x + 3)^2 + 12$

$= 12 - (x + 3)^2$

MAXIMUM and MINIMUM VALUES $a(x+b)^2+c$

For all x, $(x+b)^2 \geq 0$

expression $(x+b)^2$ has a minimum value 0 when $x = -b$

a > 0 ie. positive

for all x, $(x-3)^2 \geq 0$

$2(x-3)^2 - 8 \geq -8$

minimum value -8 when $x = 3$

a < 0 ie. negative

for all x, $-1(x+3)^2 \leq 0$

$12 - (x+3)^2 \leq 12$

maximum value 12 when $x = -3$

FRACTIONS

$\dfrac{1}{\text{min.}}$ gives max. value to a fraction

$\dfrac{1}{\text{max.}}$ gives min. value to a fraction

$$\frac{1}{3(x+2)^2 + 5}$$

for all x, $(x+2)^2 \geq 0$

$3(x+2)^2 + 5 \geq 5$

minimum value 5 when $x = -2$

$\dfrac{1}{3(x+2)^2 + 5}$	maximum value $^1/_5$ when $x = -2$

GRAPHS $y = a(x + b)^2 + c$

(i) TURNING POINT $(-b, c)$

minimum if $a > 0$ ie. positive

maximum if $a < 0$ ie. negative

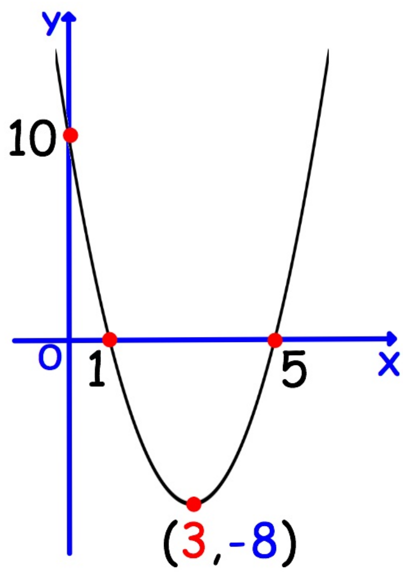
(ii) Y-INTERCEPT: substitute $x = 0$

(iii) ZEROS (if any): substitute $y = 0$,

solve $ax^2 + bx + c = 0$

factorise $(\quad)(\quad) = 0$

Sketch $y = 2x^2 - 12x + 10$



(i) $y = 2(x - 3)^2 - 8$

minimum TP: $(3, -8)$

(ii) $y = 2 \times 0^2 - 12 \times 0 + 10 = 10$

(iii) $2x^2 - 12x + 10 = 0$

$x^2 - 6x + 5 = 0$

$(x - 1)(x - 5) = 0$

$x = 1$ or $x = 5$

TRANSFORM GRAPHS

Draw the basic shape of the transformed graph.
Annotate with the images of key points.

$y = f(x) + k \qquad (x, y + k)$

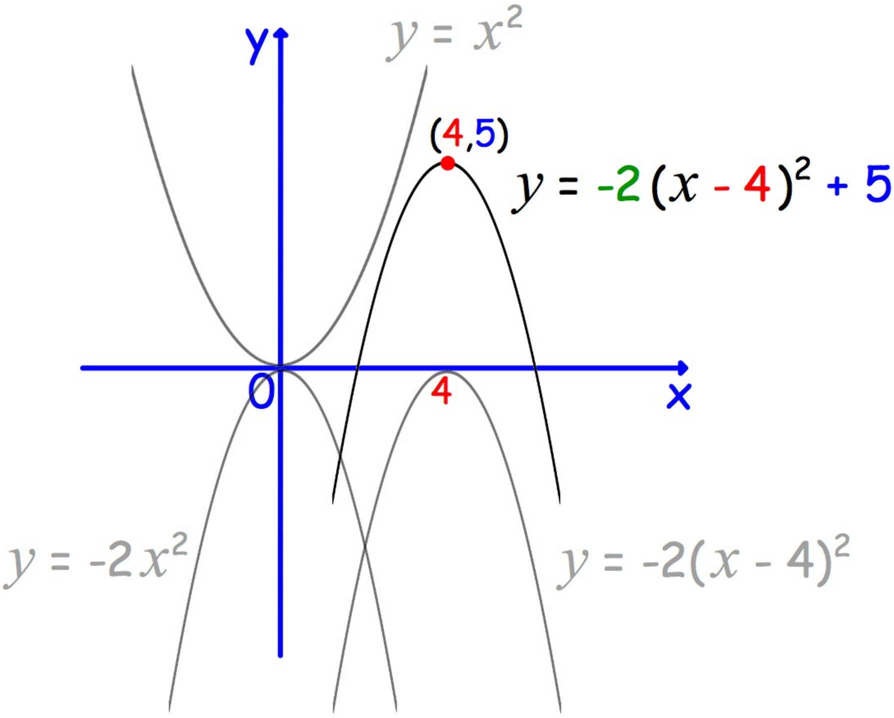
$y = f(x + k) \qquad (x - k, y)$

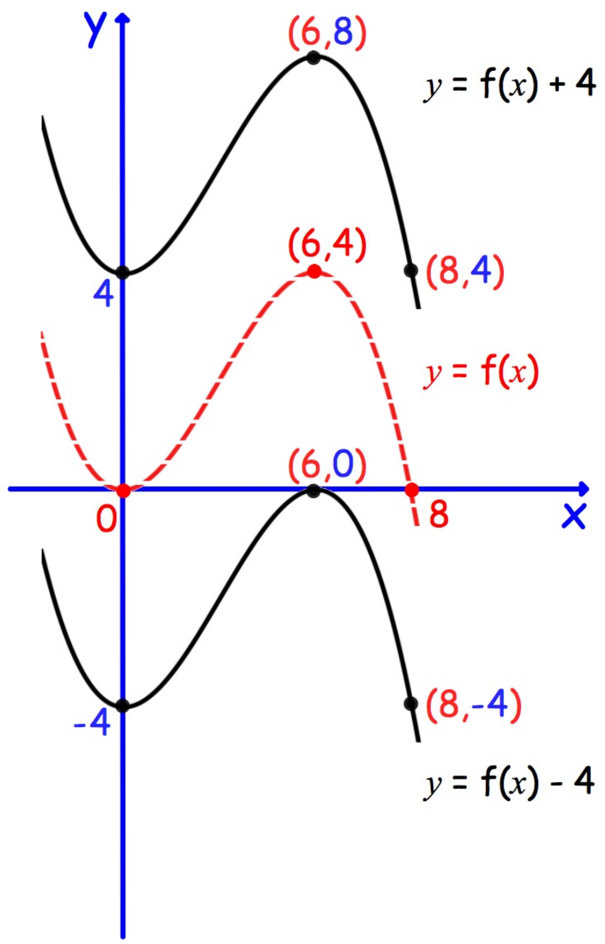
$y = kf(x) \qquad (x, ky)$

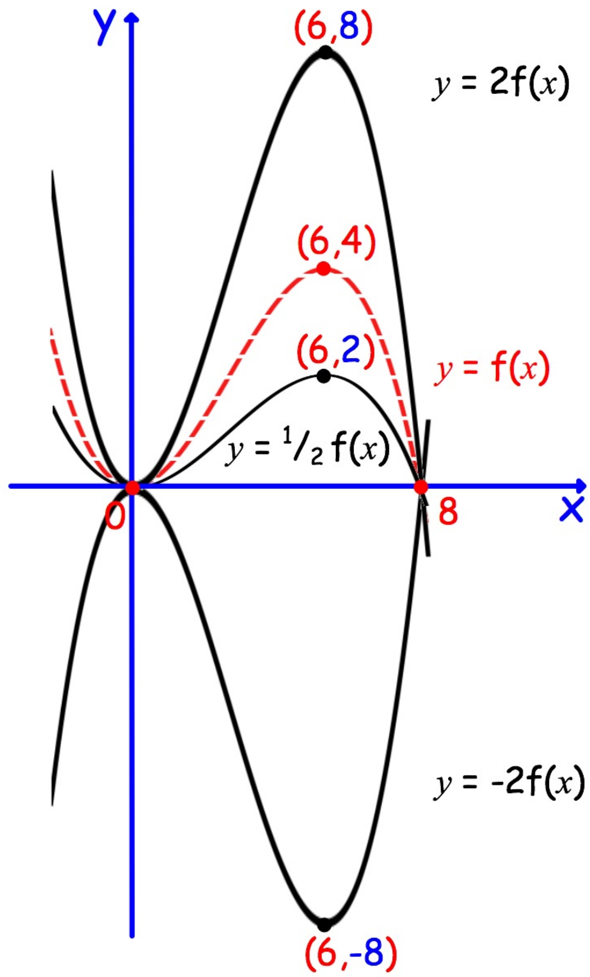
$y = f(kx) \qquad (^1/_k x, y)$

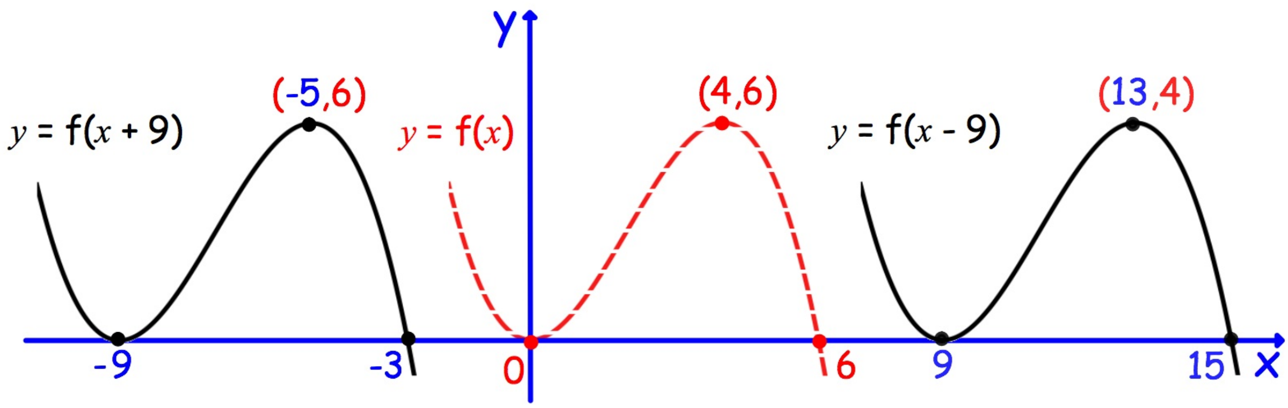
COMPLETED SQUARE

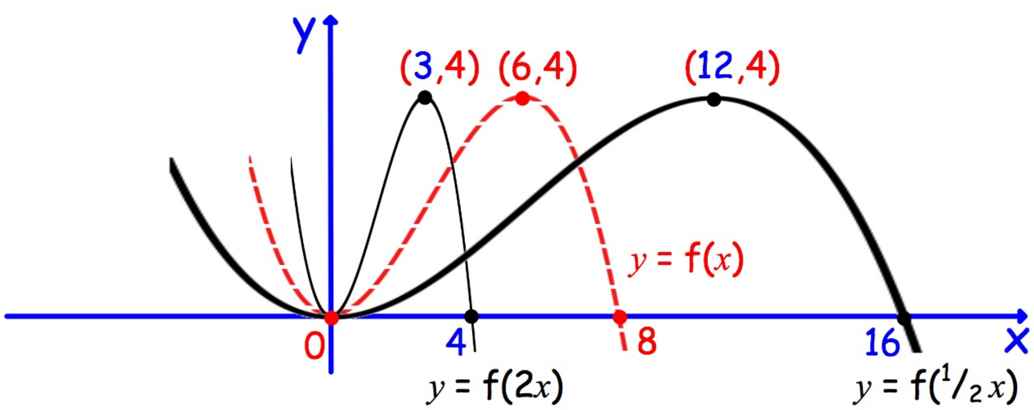
$y = 5 - 2(x - 4)^2$




y
(6,8)
y = f(x) + 4
(6,4)
(8,4)
4
y = f(x)
(6,0)
0
8
x
-4
(8,-4)
y = f(x) - 4

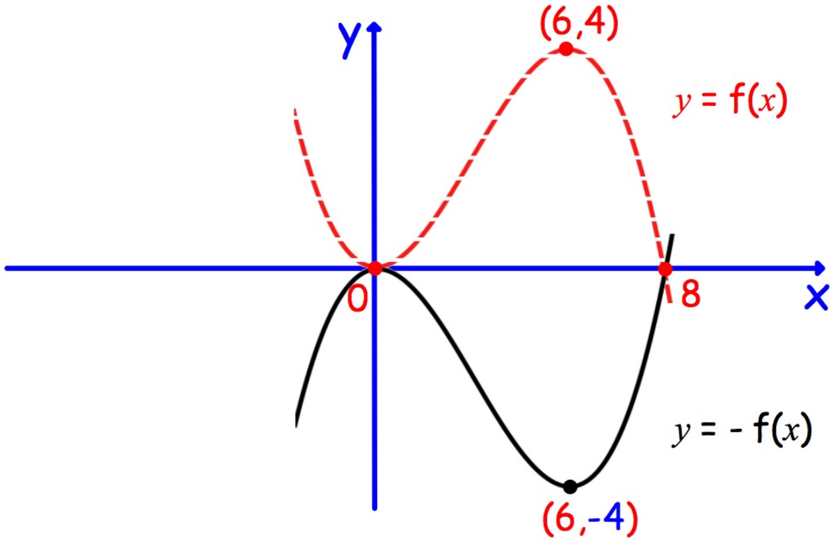

y
(6,8)
y = 2f(x)
(6,4)
(6,2)
y = f(x)
y = 1/2 f(x)
0
8
x
y = -2f(x)
(6,-8)

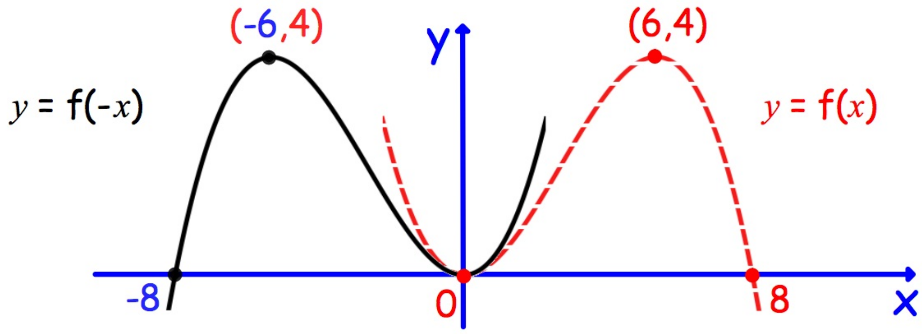

y
(-5,6)
(4,6)
(13,4)
y = f(x + 9)
y = f(x)
y = f(x - 9)
-9
-3
0
6
9
15
x

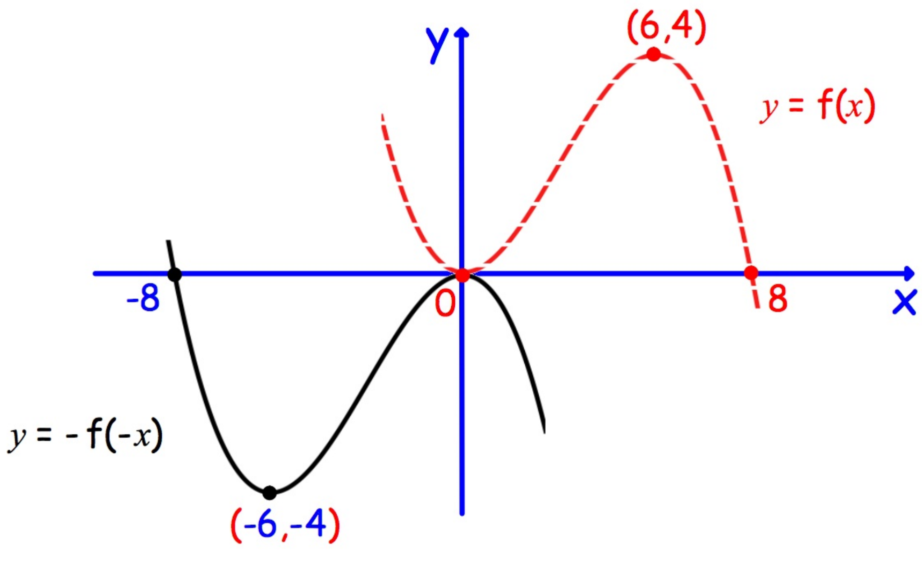

y
(3,4)
(6,4)
(12,4)
y = f(x)
0
4
8
16
x
y = f(2x)
y = f(1/2 x)

SPECIAL CASES:

REFLECT in X- axis	$y = -f(x)$	$(x, -y)$
REFLECT in Y- axis	$y = f(-x)$	$(-x, y)$
HALF-TURN about O (or REFLECT in O)	$y = -f(-x)$	$(-x, -y)$





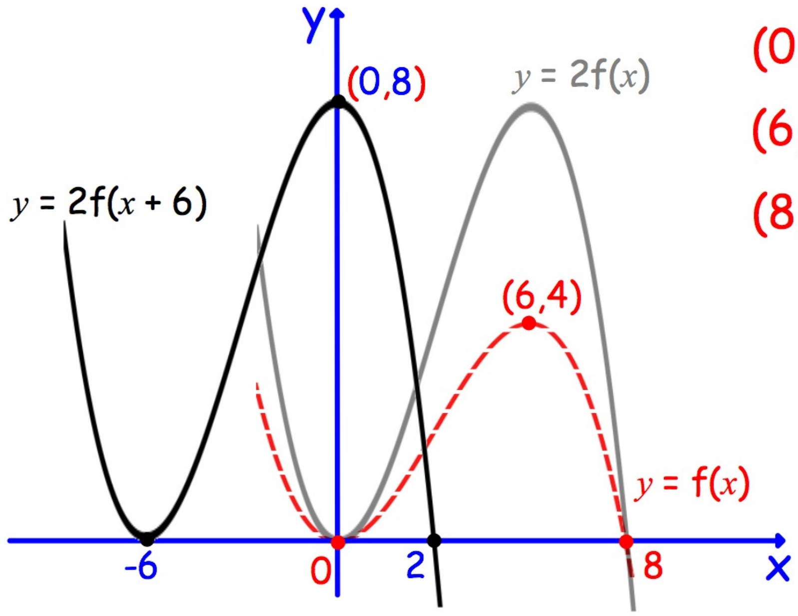


COMBINING TRANSFORMATIONS

Transform coordinates; also show intermediate graphs.

(1) $y = 2f(x+6)$

$(x, y) \longrightarrow (x-6, 2y)$



$(0,0) \longrightarrow (-6, 0)$

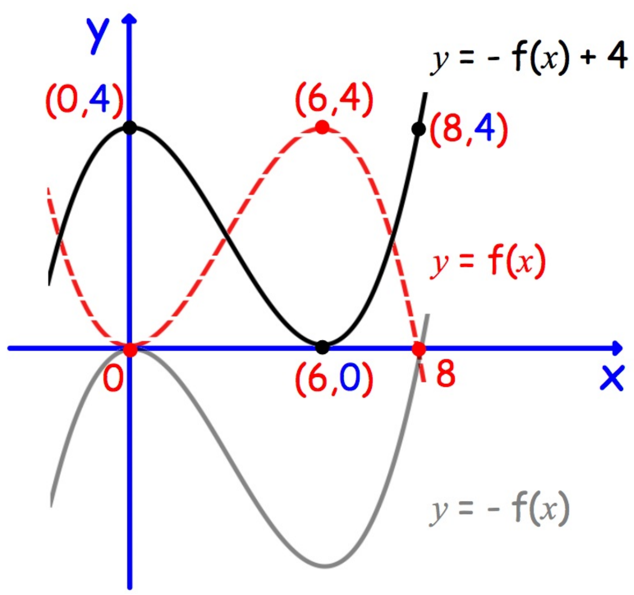
$(6,4) \longrightarrow (0, 8)$

$(8,0) \longrightarrow (2, 0)$

(2) $y = 4 - f(x)$

or $y = -f(x) + 4$

$(x, y) \longrightarrow (x, -y+4)$



$(0,0) \longrightarrow (0, 4)$

$(6,4) \longrightarrow (6, 0)$

$(8,0) \longrightarrow (8, 4)$